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Characteristic classes of Riemannian foliations

Rationality properties of the secondary classes of Riemannian foliations and some relations between the values of the classes and the geometry of Riemannian foliations are discussed.

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In memorial of
Connor Lazarov, 1937 – 2003

Born – September 26, 1937

Died – February 27, 2003

Connor was the only child of Julian and Mildred Lazarov, who were immigrants from Poland. Many of Connor's relatives, who remained in Poland, subsequently died in the Holocaust.

Connor married Sharon Rosenthal in 1970.

Connor was an avid skier and sailor.

B.A. University of Michigan 1959

Ph.D. Harvard University 1966 (advisor was Raoul Bott)

Member Institute for Advanced Studies, 1974-75

Professor of mathematics and computer science at Lehman College of CUNY 1968–2003

26 papers in major mathematics research journals (including the "Annals" and "Inventiones").

Coauthors were James Heitsch, Joel Pasternack, Herb Shulman, Al Vasquez, Arthur Wasserman

Publications by Connor Lazarov

[1] Connor Lazarov. Secondary characteristic classes in K-theory. *Trans. Amer. Math. Soc.*, 136:391–412, 1969.

[2] Connor Lazarov and Arthur Wasserman. Cobordism of regular 0(n)-manifolds. *Bull. Amer. Math. Soc.*, 75:1365–1368, 1969.

[3] C. Lazarov and A. Vasquez. The difference construction in K-theory. *Proc. Amer. Math. Soc.*, 21:315–317, 1969.

[4] Connor Lazarov and Arthur Wasserman. Cobordism of regular O(n)-manifolds. *Ann. of Math. (2)*, 93:229–251, 1971.

[5] Connor Lazarov. Actions of groups of order pq. *Trans. Amer. Math. Soc.*, 173:215–230, 1972.

[6] Connor Lazarov. Quillen's theorem for MU*. In *Proceedings of the Second Conference on Compact Transformation Groups (Univ. Massachusetts, Amherst, Mass., 1971), Part I*, pages 183–190. Lecture Notes in Math., Vol. 298, Berlin, 1972. Springer.

[7] Connor Lazarov and Arthur Wasserman. Cobordism of U(n)-actions. *Bull. Amer. Math. Soc.*, 78:1035–1038, 1972.

[8] Connor Lazarov and Arthur Wasserman. *Complex actions of Lie groups*. American Mathematical Society, Providence, R.I., 1973. Memoirs of the American Mathematical Society, No. 137.

[9] Connor Lazarov and Arthur G. Wasserman. Free actions and complex cobordism. *Proc. Amer. Math. Soc.*, 47:215–217, 1975.

[10] Connor Lazarov and Joel Pasternack. Residues and characteristic classes for Riemannian foliations. *J. Differential Geometry*, 11(4):599–612, 1976.

[11] Connor Lazarov and Joel Pasternack. Secondary characteristic classes for Riemannian foliations. *J. Differential Geometry*, 11(3):365–385, 1976.

[12] Connor Lazarov and Herbert Shulman. Obstructions to foliation preserving Lie group actions. *Topology*, 18(3):255–256, 1979.

[13] Connor Lazarov. A permanence theorem for exotic classes. *J. Differential Geom.*, 14(3):475–486 (1980), 1979.

[14] Connor Lazarov. A vanishing theorem for certain rigid classes. *Michigan Math. J.*, 28(1):89–95, 1981.

[15] Connor Lazarov and Herbert Shulman. Obstructions to foliation-preserving vector fields. *J. Pure Appl. Algebra*, 24(2):171–178, 1982.

[16] Connor Lazarov. An index theorem for foliations. *Illinois J. Math.*, 30(1):101–121, 1986.

[17] Connor Lazarov. Spectral invariants of foliations. *Michigan Math. J.*, 33(2):231–243, 1986.

[18] Connor Lazarov. A relation between index and exotic classes. In *Index theory of elliptic operators, foliations, and operator algebras (New Orleans, LA/Indianapolis, IN, 1986)*, volume 70 of *Contemp. Math.*, pages 125–143. Amer. Math. Soc., Providence, RI, 1988.

[19] James L. Heitsch and Connor Lazarov. A Lefschetz theorem for foliated manifolds. *Topology*, 29(2):127–162, 1990.

[20] James L. Heitsch and Connor Lazarov. A Lefschetz theorem on open manifolds. In *Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988)*, volume 105 of *Contemp. Math.*, pages 33–45. Amer. Math. Soc., Providence, RI, 1990.

[21] James L. Heitsch and Connor Lazarov. Homotopy invariance of foliation Betti numbers. *Invent. Math.*, 104(2):321–347, 1991.

[22] James L. Heitsch and Connor Lazarov. Rigidity theorems for foliations by surfaces and spin manifolds. *Michigan Math. J.*, 38(2):285–297, 1991.

[23] James L. Heitsch and Connor Lazarov. Spectral asymptotics of foliated manifolds. *Illinois J. Math.*, 38(4):653–678, 1994.

[24] James L. Heitsch and Connor Lazarov. A general families index theorem. *K-Theory*, 18(2):181–202, 1999.

[25] C. Lazarov. Transverse index and periodic orbits. *Geom. Funct. Anal.*, 10(1):124–159, 2000.

[26] James L. Heitsch and Connor Lazarov. Riemann-Roch-Grothendieck and torsion for foliations. *J. Geom. Anal.*, 12(3):437–468, 2002.

Characteristic Classes

Let $\mathcal{F}$ be a Riemannian foliation of smooth manifold M with tangent bundle $T\mathcal{F}$ of dimension p. Choose a Riemannian metric g on TM which is projectable when restricted to the normal bundle $\mathbf{Q} = T\mathcal{F}^{\perp}$. Let ∇_g be the basic connection on $\mathbf{Q}$ associated to this metric.

For each $1 \leq i \leq q/2$ the closed Pontrjagin form $p_i(\nabla_g) \in \Omega^{4i}(M)$ determines the Pontrjagin class $p_i \in H^{4i}(M)$.

If $q = 2n$ and $\mathbf{Q}$ is oriented, there is also the Euler class $\chi_q \in H^q(M)$ whose square $\chi_q^2 = p_n$.

The weight of an index $I = (i_1 < i_2 < \cdots < i_k)$ is

$$|I| = 4(i_1 + 2i_2 + \cdots + ki_k)$$

The degree of a monomial term $p_I = p_{i_1} \cdots p_{i_k}$ is the weight of $|I|$. In the case q is even, the degree of the monomial $\chi_q \cdot p_I$ is $|I| + q$. We let P denote a monomial of either p_I or $\chi_q \cdot p_I$ type.

THEOREM: (Pasternack) If $\deg P > q$ then

$$P(\nabla_g) = p_{i_1}(\nabla_g) \wedge \cdots \wedge p_{i_k}(\nabla_g) = 0$$

DEFINITION: If $q = 2n$, set

$$\mathbf{R}[\chi_q, p_1, p_2, \ldots, p_{n-1}]_q \equiv \mathbf{R}[\chi_q, p_1, p_2, \ldots, p_{n-1}]/\{P \mid \deg P > q\}$$

and for $q = 2n+1$, set

$$\mathbf{R}[p_1, p_2, \ldots, p_n]_q \equiv \mathbf{R}[p_1, p_2, \ldots, p_n]/\{P \mid \deg P > q\}$$

COROLLARY: (Pasternack) There is a well-defined map

$$q = 2n, \quad \Delta_* \colon \mathbf{R}[\chi_q, p_1, p_2, \ldots, p_{n-1}]_q \to H^*(M)$$

$$q = 2n+1, \quad \Delta_* \colon \mathbf{R}[p_1, p_2, \ldots, p_n]_q \to H^*(M)$$

Transgression Classes

Assume there is a trivialization $s: \epsilon^q = M \times \mathbf{R}^q \cong \mathbf{Q}$.

Let ∇_s be the associated flat connection for which s is parallel.

Set $\nabla_t = (1-t)\nabla_g + t\nabla_s$ with $p_i(\nabla_t) \in \Omega^{4i}(M \times \mathbf{R})$.

$$h_i(\nabla_g, s) = \int_0^1 p_i(\nabla_t) \in \Omega^{4i-1}(M)$$

$$dh_i(\nabla_g, s) = p_i(\nabla_g)$$

REMARK: $i > q/4 \Longrightarrow dh_i(\nabla_g, s) = 0$.

If $q = 2n$, then also set

$$h_{\chi_q}(\nabla_g, s) = \int_0^1 \chi_q(\nabla_t) \in \Omega^{q-1}(M)$$

DEFINITION: $(q = 2n)$

$$RW_q = \mathbf{R}[\chi_q, p_1, p_2, \ldots, p_{n-1}]_q \otimes \Lambda\left(h_{\chi_q}, h_1, \ldots, h_{n-1}\right)$$

$$d(1 \otimes h_{\chi_q}) = \chi_q \otimes 1, \quad d(1 \otimes p_i) = p_i \otimes 1$$

DEFINITION: $(q = 2n+1)$

$$RW_q = \mathbf{R}[p_1, p_2, \ldots, p_n]_q \otimes \Lambda\left(h_1, \ldots, h_n\right)$$

$$d(1 \otimes p_i) = p_i \otimes 1$$

DEFINITION: The basic connection ∇_g and framing s define the map of DGA's

$$\begin{array}{rcl} \Delta(\nabla_g, s): RW_q & \to & \Omega(M) \\ p_i & \to & p_i(\nabla_g, s) \\ \chi_q & \to & \chi_q(\nabla_g, s) \\ h_i & \to & h_i(\nabla_g, s) \\ h_{\chi_q} & \to & h_{\chi_q}(\nabla_g, s) \end{array}$$

Secondary Classes

THEOREM: (Lazarov) Assume that $\mathbf{Q}$ has framing s. The map on cohomology

$$\Delta_*(s)\colon H^*(RW_q) \to H^*(M)$$

is independent of the choice of basic connection ∇_g, and depends only on the homotopy class of the framing s.

THEOREM: (Lazarov) Suppose that two sections s, s' of $\mathbf{Q}$ are related by a gauge transformation $\varphi\colon M \to \mathbf{SO(q)}$,

$$s'(x) = s(x) \cdot \varphi(x), x \in M$$

Then on the level of forms,

$$\Delta(\nabla_g, s')(h_i) = \Delta(\nabla_g, s)(h_i) + \varphi^*(\tau_i)$$

where $\tau_i \in \Omega^{2i-1}(\mathbf{SO(q)})$ is the transgressive class formed from the Maurer-Cartan form. In particular, for $i > q/4$,

$$\Delta^*(s')(h_i) = \Delta^*(s)(h_i) + \varphi^*[\tau_i] \in H^{2i-1}(M)$$

Some Questions

QUESTION 1: Given a class $z \in H^*(RW_q)$, is there a Riemannian foliation of codimension q with framed normal bundle such that $\Delta^*(s)(z) \neq 0$?

QUESTION 1': Given a class $z \in H^*(RW_q)$, is there a Riemannian foliation of codimension q with framed normal bundle on a *compact* manifold M such that $\Delta^*(s)(z) \neq 0$?

QUESTION 2: Given a class $z \in H^*(RW_q)$, how does the value of $\Delta^*(s)(z) \in H^*(M)$ depend upon "the geometry and topology" of $\mathcal{F}$?

QUESTION 3: (Molino, Tokyo 1993) How do the values of the classes $z \in H^*(RW_q)$ relate to the Molino structure theory of $\mathcal{F}$?

Universal Classifying Spaces

An $R\Gamma_q$–structure on M is an open covering $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ and for each $\alpha \in \mathcal{A}$, there is given

(i) a smooth map $f_\alpha : U \to \mathbf{R}^q$
(ii) a Riemannian metric g_α on $\mathbf{R}^q$

such that the pull-backs $f_\alpha^{-1}(T\mathbf{R}^q) \to U_\alpha$ define a vector bundle $\mathbf{Q} \to M$ with Riemannian metric $g \mid \mathbf{Q} = g_\alpha = f_\alpha^* g_\alpha$

THEOREM: (Haefliger) A Riemannian foliation $\mathcal{F}$ on M defines an $R\Gamma_q$–structure on M. The homotopy class of the composition

$$h_{\mathcal{F}} : M \cong B\mathcal{U} \to BR\Gamma_q$$

depends only on the integrable homotopy class of $\mathcal{F}$.

The normal bundle to the universal $R\Gamma_q$–structure on $BR\Gamma_q$ admits a classifying map $\nu : BR\Gamma_q \to B\mathbf{O(q)}$. The homotopy fiber of ν is the space $B\overline{R\Gamma}_q$ This classifies $R\Gamma_q$–structures with a (homotopy class of) framing for $\mathbf{Q}$.

$$\begin{array}{ccccccc}
\mathbf{O(q)} & = & \mathbf{O(q)} & \simeq & \Omega B\mathbf{O(q)} \\
\downarrow & & \downarrow & & \downarrow \\
P & \overset{\overline{h_{\mathcal{F},s}}}{\longrightarrow} & FR\Gamma_q & \simeq & B\overline{R\Gamma}_q \\
s \uparrow\downarrow & & \downarrow & & \downarrow \\
M & \overset{h_{\mathcal{F}}}{\longrightarrow} & BR\Gamma_q & = & BR\Gamma_q \\
 & & \downarrow\ \nu & & \downarrow \nu \\
 & & B\mathbf{O(q)} & = & B\mathbf{O(q)}
\end{array}$$

QUESTION 4: What is the homotopy type of $BR\Gamma_q$? of $B\overline{R\Gamma}_q$?

Homotopy Type of $B\overline{R\Gamma}_q$

THEOREM: (Pasternack) $B\overline{R\Gamma}_1 \simeq B\mathbf{R}^\delta$

THEOREM: (Lazarov) If $q = 4k-2, 4k-1$, then

$$\pi_{4k-1}(BR\Gamma_q) \to \mathbf{R}^\ell \to \mathbf{0}, \quad \ell > 0$$

THEOREM: (Hurder) $\nu\colon BR\Gamma_q \to B\mathbf{O(q)}$ is a q-connected map. That is, $B\overline{R\Gamma}_q$ is $q-1$-connected.

<u>Proof:</u> Milnor's remark $\Longrightarrow$ by the Phillips immersion theorem, an $\overline{R\Gamma}_q$–structure on $\mathbf{S}^\ell$ for $0 < \ell < q$ corresponds to a Riemannian metric defined on an open neighborhood retract $\mathbf{S}^\ell \subset U \subset \mathbf{R}^q$.

An $\overline{R\Gamma}_q$–structure on an open set in $\mathbf{R}^q$ is explicitly homotopic to the Euclidean metric on an open neighborhood retract $\mathbf{S}^\ell \subset V \subset \mathbf{R}^q$, which is the "trivial" $\overline{R\Gamma}_q$–structure.

Thus, the given $\overline{R\Gamma}_q$–structure on $\mathbf{S}^\ell$ can be homotoped to a trivial structure, hence represents the trivial $\overline{R\Gamma}_q$–structure on $\mathbf{S}^\ell$.

QUESTION 5: Is $B\overline{R\Gamma}_q$ q-connected for $q \neq 4k-1$?

QUESTION 5': What are the integrable homotopy invariants of a 1–dimensional Riemannian foliation on an open manifold?

Injectivity of the Pontrjagin Classes

THEOREM: (Thom) There is a compact, orientable Riemannian manifold B of dimension q such that all of the Pontrjagin and Euler classes up to degree q are independent in $H^*(B)$.

If q is odd, then B can be chosen to be a connected manifold.

Proof: For q even, B is the disjoint union of all products of the form

$$\mathbf{CP}^{i_1} \times \cdots \times \mathbf{CP}^{i_k} \times S^1 \times \cdots \times S^1$$

with dimension q.

For q odd, B is the connected sum of all products of the form

$$\mathbf{CP}^{i_1} \times \cdots \times \mathbf{CP}^{i_k} \times S^1 \times \cdots \times S^1$$

with dimension q.

Injectivity of the Secondary Classes

THEOREM: (Hurder) There exists a compact manifold M and a Riemannian foliation $\mathcal{F}$ on M with trivial normal bundle, such that $\mathcal{F}$ is defined by a fibration over a compact manifold of dimension q, and the secondary characteristic map injects:

$$\Delta_*(s) \colon H^*(RW_q) \hookrightarrow H^*(M)$$

Moreover, if q is odd, then M can be chosen to be connected.

Proof: Let B be the compact, orientable Riemannian manifold defined in the proof of Thom's Theorem.

Let M be the bundle of oriented orthonormal frames for TB.

The basepoint map $\pi \colon M \to B$ defines a fibration

$$\mathbf{SO(q)} \to M \to B$$

with fiber over $x \in B$ the group $\mathbf{SO(q)}$ of orthonormal frames in $T_x B$.

$\mathcal{F}$ is the foliation defined by the fibration. The Riemannian metric on B lifts to the transverse metric on the normal bundle $\mathbf{Q} = \pi^{-1}TB$. s is the canonical framing of $\mathbf{Q}$.

The basic connection ∇_g and framing s define the map of DGA's

$$\Delta(\nabla_g, s) \colon RW_q \to \Omega(M)$$

Consider a fiber $L_x = \pi^{-1}(x)$ – the normal bundle restricted to L_x is trivial, as it is just the constant lift of $T_x B$. Thus, the basic connection ∇_g is the connection associated to the product bundle.

The connection ∇_s for which the canonical framing is parallel, equals the connection defined by the Maurer-Cartan form on $\mathbf{SO(q)}$.

By Chern-Weil theory, the forms $h_i(\nabla_g, s)$ and $h_{\chi_q}(\nabla_g, s)$ restricted to L_x are closed, and their classes in cohomology define free exterior generators for the cohomology $H^*(\mathbf{SO(q)})$.

The characteristic map on forms

$$\Delta(\nabla_g, s) \colon RW_q \to \Omega(M)$$

induces a map of the basic spectral sequence,

$$\Delta_p^{*,*} \colon E_2^{*,*}(RW_q) \to E_p^{*,*}(M)$$

At the E_2 stage, this is

$$\Delta_2^{*,*} \colon RW_q \to H^*(\mathbf{SO(q)}) \otimes H^*(B)$$

which is injective by construction.

Pass to the E_∞–limit to obtain that

$$\Delta^*(s) \colon H^*(RW_q) \to H^*(M)$$

induces an injective map of associated graded algebras, hence it is also injective.

Note that in this proof:

- All leaves are compact.
- The "geometry" of $\mathcal{F}$ (seems) to make no difference.
- The image of the "basis" classes are integral

$\Delta^*(s)(h_J p_I) = \Delta^*(s)(h_{j_1} \wedge \cdots \wedge h_{i_\ell} \cdot p_{i_1} \cdots p_{i_k}) \in \{H^*(M, \mathbf{R}) \leftarrow H^*(M, \mathbf{Z})\}$

QUESTION 1": Does there exists a compact connected manifold M and a Riemannian foliation $\mathcal{F}$ with even codimension and trivial normal bundle, such that the secondary characteristic map $\Delta_*(s)$ injects?

Variable and Rigid Classes

The secondary classes of a foliation are divided into two types, variable and rigid.

The variable classes are most sensitive to geometry and dynamics.

The rigid classes seem to be topological in nature.

A basis element

$$h_J p_I = h_{j_1} \wedge \cdots \wedge h_{i_\ell} \cdot p_{i_1} \cdots p_{i_k}$$

is a rigid class if it lies in the image of

$$H^*(RW_{q+1}) \to H^*(RW_q)$$

The variable classes are the cokernel of this map.

A rigid class is invariant for a deformation $\mathcal{F}_t$, $0 \leq t \leq 1$ through Riemannian foliations: The family $\mathcal{F}_t$ defines a Riemannian foliation $\mathcal{F} \times [0,1]$ on $M \times [0,1]$ of codimension $q+1$, and the rigid classes are defined for $\mathcal{F} \times [0,1]$ so restrict to the same class at the ends.

The variable classes do not have this property, so *a priori* may vary non-trivially under deformation.

The variable classes exist only when $q = 4k-2$ or $q = 4k-1$.

A spanning set for the variable classes is obtained by considering all terms

$$h_{j_1} \cdot p_{i_1} \cdots p_{i_k} \in RW_q$$

of degree $4k-1$, where $j_1 \leq i_\ell$ for all ℓ. We then extend this set to include all terms $h_{j_1} \wedge \cdots \wedge h_{i_\ell} \cdot p_{i_1} \cdots p_{i_k}$ where $j_\ell > j_i$ for $\ell > 1$.

Integrality

The integrality of the secondary classes in the proof of the non-triviality theorem illustrates a general property of the secondary classes:

THEOREM: (Hurder) Let $\mathcal{F}$ be a Riemannian foliation of codimension q on M with trivial normal bundle. Suppose there exists a manifold B of dimension q and a submersion $\pi\colon M \to B$ whose fibers define $\mathcal{F}$.

If $q = 2n$ is even, then the image of the basis classes $h_J p_I$ are rational

$$\Delta^*(s)(h_J p_I) \in \mathbf{Image}\,\{H^*(M, \mathbf{Q}) \to H^*(M, \mathbf{R})\}$$

When $\deg p_I = q$, the image of $\Delta^*(s)(h_J p_I)$ lies in the image of the integral cohomology.

If $q = 2n + 1$ is odd, then the image of the rigid basis classes are rational

$$\Delta^*(s)(h_J p_I) \in \mathbf{Image}\,\{H^*(M, \mathbf{Q}) \to H^*(M, \mathbf{R})\}$$

When $\deg p_I = 2n + 2$, the image of $\Delta^*(s)(h_J p_I)$ lies in the image of the integral cohomology.

Proof: Use standard rational homotopy techniques and functoriality of the characteristic maps to show that an integral (or rational) model of RW_q factors through the rational de Rham complex of M.

The restriction on the types of classes for q odd is essential. The proof uses "rigidity" properties of the classes, but applied within the level of DGA's and not to a geometric deformation of the foliation.

Compact Hausdorff Foliations

DEFINITION: A foliation of a manifold M is *compact Hausdorff* if every leaf of $\mathcal{F}$ is a compact manifold, and the leaf space $M/\mathcal{F}$ is a Hausdorff space.

THEOREM: (Epstein, Millet) A compact Hausdorff foliation $\mathcal{F}$ is Riemannian – there exists a transverse holonomy invariant Riemannian metric on the normal bundle $\mathbf{Q}$ for $\mathcal{F}$.

THEOREM: (Hurder) Let $\mathcal{F}$ be a compact Hausdorff foliation of codimension q on M with trivial normal bundle. If $h_J p_I$ is a rigid class, or if q is even and $h_J p_I$ is any basis element, then

$$\Delta^*(s)(h_J p_I) \in \mathbf{Image}\,\{H^*(M, \mathbf{Q}) \to H^*(M, \mathbf{R})\}$$

Super-rigidity and Riemannian Foliations

THEOREM: Suppose that M is a compact manifold with fundamental group Γ. Assume that Γ is an irreducible lattice of real-rank r. (e.g., $\Gamma = \mathbf{SL}(r+1, \mathbf{Z})$) Then every Riemannian foliation of M of codimension $q < r$ is compact Hausdorff.

When the codimension $q > r$ where r is the real-rank of Γ, then Dupont and Kamber have obtained rationality results for certain of the secondary classes, when $\mathcal{F}$ is a point foliation. See "Cheeger-Chern-Simons classes of transversally symmetric foliations," Math. Ann. 295 (1993)

Continuous Variation

Consider $\mathbf{S}^3$ as the Lie group $SU(2)$ with Lie algebra spanned by

$$X = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

.

THEOREM: (Chern-Simons) Let g_u be the Riemannian metric on $\mathbf{S}^3$ where the parallel Lie vector fields $\{u \cdot X, Y, Z\}$ is an orthonormal basis. Then for $\mathcal{F}_u$ the point-foliation on $\mathbf{S}^3$ with transverse metric g_u, $\Delta^*(s)(h_1) \in H^3(M)$ is a non-constant function of u.

THEOREM: (Lazarov) Let $q = 4k - 2$. Then for every non-zero $z \in H^{4k-1}(RW_q)$, there is a family of Riemannian foliations $\mathcal{F}_t$ with trivial normal bundle on a compact manifold M such that the images of the class $\Delta(s)(z) \in H^{4k-1}(M)$ is a non-constant continuous function of t.

Lazarov proved much more – he showed the complete independent variation of all classes in these degrees.

THEOREM: (Lazarov) Let $q = 4k-2$. The universal characteristic map

$$\Delta^* \colon H^{q+1}(RW_q) \to H^{q+1}(B\overline{R\Gamma}_q)$$

is injective, and all classes in the image "vary independently".

The geometry of deformation examples involves either:

• For $q = 4k-1$, a deformation of the transverse Riemannian metric on $\mathbf{Q}$. This is analyzed via Cheeger-Chern-Simons' theory.

• For $q = 4k - 2$, a geometric deformation of the foliation $\mathcal{F}_t$ in a neighborhood of a singular set. This is analyzed via residue theory.

Molino Theory

Let $\mathcal{F}$ be a Riemannian foliation of codimension q on a compact manifold M, with oriented normal bundle $\mathbf{Q}$.

Let $\pi\colon P \to M$ be the bundle of oriented orthonormal frames in $\mathbf{Q}$, with fiber group $\mathbf{SO(q)}$.

The foliation $\mathcal{F}$ is covered by a Riemannian foliation $\widehat{\mathcal{F}}$ on P of the same dimension.

THEOREM: (Molino) The closures of the leaves of $\widehat{\mathcal{F}}$ define a Riemannian foliation $\mathcal{G} = \overline{\widehat{\mathcal{F}}}$ of P with all leaves compact and no holonomy. Thus, $\mathcal{G}$ is defined by a fibration $P \to W$.

As $\widehat{\mathcal{F}}$ is $\mathbf{SO(q)}$-invariant, the foliation $\mathcal{G}$ and is also $\mathbf{SO(q)}$-invariant, W is a $\mathbf{SO(q)}$-space, and the quotient map $P \to W$ is $\mathbf{SO(q)}$-equivariant.

For a leaf $\widehat{L}$ of $\widehat{\mathcal{F}}$, the restriction of $\widehat{\mathcal{F}}$ to the closure $\overline{\widehat{L}}$ in P is a Lie G-foliation with all leaves dense. That is, for each $x \in P$ there is a connected Lie group G_x, a complete manifold Z_x with fundamental group $\Gamma_x = \pi(Z_x, x)$ and universal cover $\widetilde{Z}_x$, and representation $\rho_x\colon \Gamma_x \to G_x$ so that the quotient space $(\widetilde{Z}_x \times G_x)/\Gamma_x \cong \overline{\widehat{L}}$ as foliated manifolds. The identification is given by a natural map induced from the Lie algebra of G_x realized as basic vector fields for $\widehat{\mathcal{F}}$.

The for a leaf $\widehat{L} \subset P$ which covers a leaf $L \subset M$, $\pi(\overline{\widehat{L}}) = \overline{L}$.

Let F_x denote the fiber of $\pi\colon M \to B$ containing x. We identify F_x with $\mathbf{SO(q)}$ where x corresponds to the identity element. Then $H_x = \widehat{L} \cap F_x$ is a closed subspace of F_x, and is identified with a subgroup of $\mathbf{SO(q)}$.

The set of leaf closures $\overline{L}$ for $\mathcal{F}$ define a singular foliation $\overline{\mathcal{F}}$ on M.

Plus more...

Regular Riemannian Foliations

DEFINITION: A Riemannian foliation of a compact manifold M is *regular* if the set of leaf closures form a non-singular foliation $\overline{\mathcal{F}}$.

EXAMPLE: A Riemannian foliation with every leaf compact is regular. These are the compact Hausdorff foliations.

Assume that M is compact and the normal bundle $\mathbf{Q}$ to $\mathcal{F}$ is framed.

Let L be a leaf of $\mathcal{F}$ with closure $\overline{L}$.

Let $\widehat{L}$ a leaf of $\widehat{\mathcal{F}}$ which covers L.

The Molino structure theory gives $\overline{\widehat{L}} \cong (\widetilde{Z}_x \times G_x)/\Gamma_x$ so

$$\overline{L} \cong (\widetilde{Z}_x \times G_x/H_x)/\Gamma_x$$

where $H_x \subset G_x$ is a closed subgroup.

The normal bundle of $\mathcal{F}$ restricted to $\overline{L}$ is identified with the bundle associated to the adjoint representation of Γ_x on $m = T_x(G_x/H_x)$

The restriction of $\mathbf{Q}$ to $\overline{L}$ decomposes into $\mathbf{Q} = \mathbf{Q}_1 \oplus \mathbf{Q}_2$ where $\mathbf{Q}_1$ is tangent to $\overline{L}$ and $\mathbf{Q}_2$ is orthogonal to $\overline{L}$. Let q_2 denote the codimension of $\overline{\mathcal{F}}$, hence equal to the dimension of $\mathbf{Q}_2$.

LEMMA: If $\mathcal{F}$ is regular, then $\mathbf{Q}_2$ is finitely covered by a trivial bundle.

Proof: $\overline{\mathcal{F}}$ is a Riemannian foliation with all leaves compact, hence is compact Hausdorff. The holonomy of each leaf $\overline{L}$ of $\overline{\mathcal{F}}$ is finite, so there is a finite covering of $\overline{L}$ on which the restriction of $\mathbf{Q}_2$ is trivial.

COROLLARY: The Pontrjagin forms of $\mathbf{Q}$ vanish in degrees greater than the codimension q_2 of $\overline{\mathcal{F}}$.

Rationality and Vanishing for Regular Foliations

THEOREM: Let $\mathcal{F}$ be a regular Riemannian foliation of codimension q on M with trivial normal bundle. Suppose that the leaf closures $\overline{\mathcal{F}}$ define a foliation of codimension $q_2 < q$. Then $\Delta^*(s)(h_J p_I) = 0 \in H^*(M, \mathbf{R})$ if $\deg p_I > q_2$.

THEOREM: Let $\mathcal{F}$ be a regular Riemannian foliation of codimension q on M with trivial normal bundle. If $h_J p_I$ is a rigid class, or if q is even and $h_J p_I$ is any basis element, then

$$\Delta^*(s)(h_J p_I) \in \mathbf{Image}\,\{H^*(M, \mathbf{Q}) \to H^*(M, \mathbf{R})\}$$

The requirement on dimension and rigid class is required, as the following example shows.

EXAMPLE: Let $\mathbf{S}^3$ have the point foliation $\mathcal{F}_u$ with transverse metric g_u such that $\Delta^*(s)(h_1) \in H^3(M)$ is a non-constant function of u.

Let B be a compact oriented Riemannian manifold of dimension 8 such that $p_2(TB) \neq 0$. Let $P \to B$ be the bundle of oriented frames for TB. Then P has a codimension 8 foliation $\mathcal{F}'$ with trivial normal bundle for which $\Delta_*(h_2 \cdot p_2) \neq 0 \in H^{15}(P)$.

The product manifold $V = \mathbf{S}^3 \times P$ has a family of Riemannian foliations $\mathcal{F}''_u = \mathcal{F}_u \times \mathcal{F}'$ of codimension 11, whose normal bundle is framed by the sum of the framings of the normal bundles on each factor. Then the class $\Delta_*(s)(h_1 \wedge h_2 \cdot p_2) \in H^{18}(V)$ and is a non-constant function of u.

Note that $h_1 \cdot p_2$ is a variable class, so also is $h_1 \wedge h_2 \cdot p_2$.

Vanishing for Regular Foliations

The last result is a vanishing theorem for all of the secondary classes if the topology of M is restricted.

THEOREM: Let $\mathcal{F}$ be a regular Riemannian foliation of codimension q on M with trivial normal bundle. Suppose that $\pi_1(M)$ is finite, and the leaf closures $\overline{\mathcal{F}}$ define a foliation of codimension $q_2 < q$. Then $\Delta^*(s)(h_J p_I) = 0 \in H^m(M, \mathbf{R})$ where M has dimension m. That is, all of the secondary characteristic numbers for $\mathcal{F}$ vanish.