CLASSIFYING SPACES FOR GROUPOID STRUCTURES

TAKASHI TSUBOI

Abstract. This is an introductory survey on the theory of classifying spaces for the groupoid structures.

The most important groupoid is the groupoid Γ_q of germs of local diffeomorphisms of $\mathbf{R}^q$. Then the classifying space $B\Gamma_q$ classifies the foliations of codimension q.

We review the groupoid of germs of local diffeomorphisms, construction of the classifying space for the groupoid, foliated microbundles, and some related notions. We state the Mather-Thurston isomorphism theorem: $H_*(B\overline{\mathrm{Diff}}{}^r_c(\mathbf{R}^n);\mathbf{Z}) \cong H_*(\Omega^n B\overline{\Gamma}{}^r_n;\mathbf{Z})$, and explain its meaning.

We state the theory of existence of foliations by Gromov-Phillips-Haefliger-Thurston and give a list of known results on the topology of $B\Gamma_q$. The topology of $B\Gamma_q$ turned out to depend on the regularity (differentiability).

We also review the characterization by Segal-Haefliger of the classifying space for a groupoid structure. This is stated as follows: $(X,\mathcal{F}) \simeq B\Gamma_{\mathcal{F}}$ if and only if the holonomy covering of any leaf is contractible. If we call such foliation typical, the Reeb foliation is typical. This gives rise a series of question.

1. Groupoid structures

Let $\mathrm{Diff}(M)$ be the group of C^∞ diffeomorphisms of a manifold M. Let Γ_M denote the set of germs of elements of $\mathrm{Diff}(M)$. That is an element of Γ_M is the equivalence class $[f,x]$ of (f,x), where $f \in \mathrm{Diff}(M)$, $x \in M$, and $(f_0,x_0) \sim (f_1,x_1)$ if and only if $x_0 = x_1$ and there is a neighbourhood U of x_0 such that $f_0|U = f_1|U$. There are well defined maps s (source) and t (target) : $\Gamma_M \longrightarrow M$ defined by $s([f,x]) = x$ and $t([f,x]) = f(x)$. There is a map e in the other direction $e : M \longrightarrow \Gamma_M$ defined by $e(x) = [\mathrm{id}, x]$. There is a topology on Γ_M defined so that a basis of the neighborhoods of $[f,x]$ is given by the set consisting of sets $\{[f,y]; y \in U\}$ where (f,x) are representatives and U are neighborhoods of x. Then the maps s, t, e are continuous.

Definition 1.1. A topological groupoid is a small topological category such that all morphisms are invertible.

Here "small category" means there are the set Obj of objects, the set Mor of morphisms, and maps $s, t : Mor \longrightarrow Obj$, $e : Obj \longrightarrow Mor$ with appropriate properties. "Topological" means s, t, e, the composition and the inversion are continuous. For a morphism γ, it makes the life easier if we write an arrow as

$$t(\gamma) \xleftarrow{\gamma} s(\gamma) \quad .$$

Since the set of objects are identified with the set of identity morphisms, the set $\Gamma = Mor$ of morphisms stands for the groupoid.

Example 1.2. $Obj = M$, $Mor = \Gamma_M$.

$$f(x) \xleftarrow{[f,x]} x$$

Example 1.3 (Covering groupoid)**.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of M. Put $Obj = \bigsqcup U_i$ and $Mor = \Gamma_{\mathcal{U}} = \{(x, i, j)\ ;\ i, j \in I,\ x \in U_i \cap U_j \subset M$.

$$(x \in U_i) \xleftarrow{(x,i,j)} (x \in U_j)$$

The composition is defined as $(x, i, j)(x, j, k) = (x, i, k)$.

Quiz 1.4. For a set E, we have the binal (pair, coarse) groupoid defined by $Obj = E$, $Mor = E \times E$, $i \xleftarrow{(i,j)} j$ and $(i, j)(j, k) = (i, k)$. Then a "kernel" of the homomorphism (functor) $\Gamma_{\mathcal{U}} \longrightarrow M$ is a binal groupoid.

Definition 1.5. For a (topological) groupoid Γ, a Γ structure of $(M, \mathcal{U})$ is a (topological) groupoid homomorphism (a functor) $\Phi : \Gamma_{\mathcal{U}} \longrightarrow \Gamma$.

Then it is natural to define a Γ structure on M to be an equivalence class of homomorphisms $\Phi : \Gamma_{\mathcal{U}} \longrightarrow \Gamma$, where the homomorphisms are equivalent if they induces the same homomorphism for a refined open covering.

Quiz 1.6. A continuous map $f : M \longrightarrow N$ induces a functor between covering groupoids. Hence a Γ structure $\mathcal{H}$ on N induces the pullback $f^*\mathcal{H}$ on M.

We will see that a Γ-structure $\Phi : \Gamma_{\mathcal{U}} \longrightarrow \Gamma$ induces the "classifying" map $B\Phi : B\Gamma_{\mathcal{U}} \longrightarrow B\Gamma$, where $B\Gamma_{\mathcal{U}}$ is homotopy equivalent to M.

2. Foliations and groupoid structures

Let Γ_q^r denote the groupoid of germs of C^r diffeomorphisms of $\mathbf{R}^q$. By definition a Γ_q^r structure of $(M, \mathcal{U})$ is a homomorphism $\Phi : \Gamma_{\mathcal{U}} \longrightarrow \Gamma_q^r$.

If we write down the objects of groupoids, it is given by a family of (structural) continuous maps.

$$\begin{array}{ccc} \Gamma_{\mathcal{U}} & \xrightarrow{\Phi} & \Gamma_q^r \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ \coprod U_i & \xrightarrow[f]{} & \mathbf{R}^q \end{array} .$$

Put $\gamma_{ij}^x = \Phi(x,i,j)$ and $f|U_i = f_i$. In the covering groupoid, we have

$$(x \in U_i) \xleftarrow{(x,i,j)} (x \in U_j)$$

and the image of this morphism is

$$f_i(x) \xleftarrow{\gamma_{ij}^x} f_j(x).$$

The usual definition of Γ_q^r structure is given by $\{U_i, f_i, \gamma_{ij}^x\}_{i,j\in I}$ satisfying the cocycle condition $\gamma_{ij}^x\gamma_{jk}^x = \gamma_{ik}^x$. The cocycle condition is just the functoriality of Φ.

Now the definition of a foliation is as follows.

Definition 2.1. A codimension q, C^r foliation of M is a Γ_q^r structure such that $f_i : U_i \longrightarrow \mathbf{R}^q$ are the submersions.

Submersions are differentiable maps with Jacobians being maximal rank (usually $= q$).

3. Classifying spaces for groupoid structures

Now we explain how to associate a space for a topological groupoid. We follow Segal's fat realization construction. Another method is Milnor's infinite join construction.

For a groupoid Γ, put $\Gamma^{(0)}$ be the set of objects X or the set of identity morphisms in Γ, $\Gamma^{(1)} = \Gamma$, $\Gamma^{(n)}$ be the n-tuple $(\gamma_1, \dots, \gamma_n)$ such that γ_{i-1} and γ_i are composable ($s(\gamma_{i-1}) = t(\gamma_i)$). It is worth thinking the following sequence as an element of $\Gamma^{(n)}$:

$$x_0 \xleftarrow{\gamma_1} x_1 \xleftarrow{\gamma_2} x_2 \longleftarrow \cdots \longleftarrow x_{n-1} \xleftarrow{\gamma_n} x_n.$$

These $\Gamma^{(n)}$ are related by the face maps ∂_i.

$$\Gamma^{(0)} \;\substack{\longleftarrow \\ \longleftarrow}\; \Gamma^{(1)} \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \Gamma^{(2)} \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \Gamma^{(3)} \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}$$

For $\gamma \in \Gamma^{(1)} = \Gamma$, put $\partial_0(\gamma) = s(\gamma)$ and $\partial_1(\gamma) = t(\gamma)$. We define the face maps for $n \geq 2$ as follows:

$$\begin{aligned}
\partial_i(\gamma_1, \ldots, \gamma_n) &= \ (\gamma_1, \ldots, \gamma_i\gamma_{i+1}, \ldots, \gamma_n) \quad (i = 1,\ \ldots,\ n) \\
\partial_0(\gamma_1, \ldots, \gamma_n) &= \ (\gamma_2, \ldots, \gamma_n) \\
\partial_n(\gamma_1, \ldots, \gamma_n) &= \ (\gamma_1, \ldots, \gamma_{n-1})
\end{aligned}$$

Then the space $B\Gamma$ is defined as follows.

Definition 3.1.

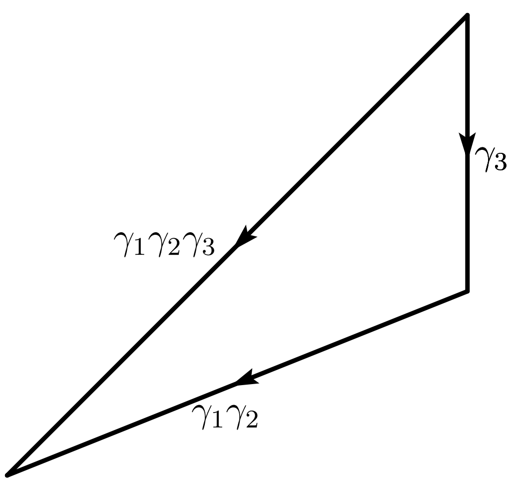
$$B\Gamma = \bigsqcup_n \Gamma^{(n)} \times \triangle^n / \sim,$$

where $\triangle^n$ is the standard n simplex. By using the i-th face map $\delta_i : \triangle^{n-1} \longrightarrow \triangle^n$ $(i = 0, \ \ldots\ , n)$, the equivalence $\sim$ is defined by

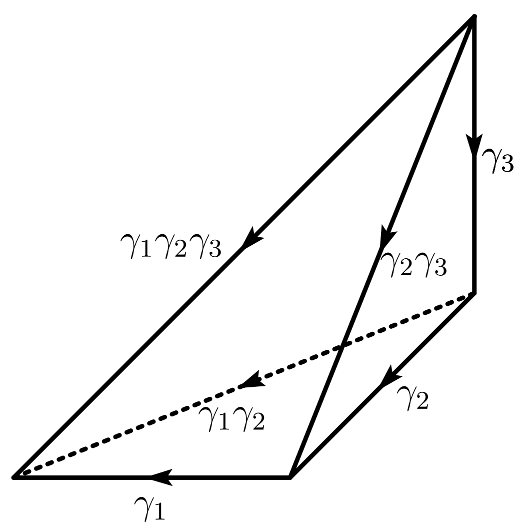
$$\Gamma^{(n-1)} \times \triangle^{n-1} \ni \partial_i\sigma \times u \sim \sigma \times \delta_i u \in \Gamma^{(n)} \times \triangle^n$$

for $\sigma \in \Gamma^{(n)}$ and $u \in \triangle^{n-1}$.

The points of $(\gamma_1, \ldots, \gamma_n) \times \triangle^n$ are understood as the points in the simplex with longest sequence of edges being named as $(\gamma_1, \ldots, \gamma_n)$.



$\partial_1(\gamma_1, \gamma_2, \gamma_3) \times \triangle^2$
$= (\gamma_1\gamma_2, \gamma_3) \times \triangle^2$



$(\gamma_1, \gamma_2, \gamma_3) \times \triangle^3$

The topology of the space $B\Gamma$ is the quotient topology induced from the topology of $\bigsqcup \Gamma^{(n)} \times \triangle^n$. (We recommend not to seriously think about it.)

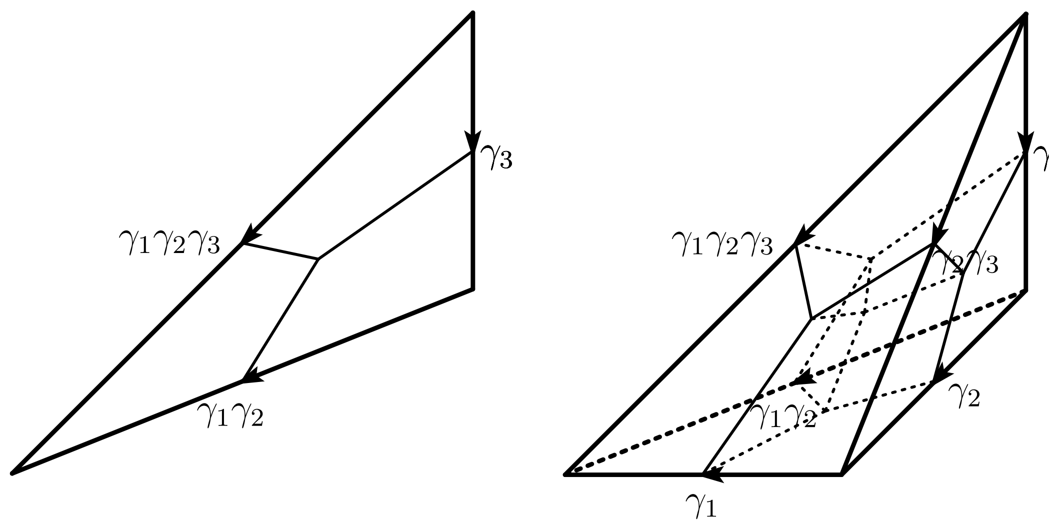
Now for the homomorphism $\Phi : \Gamma_{\mathcal{U}} \longrightarrow \Gamma$, we obtain the induced map $B\Phi : B\Gamma_{\mathcal{U}} \longrightarrow B\Gamma$. We notice that $B\Gamma_{\mathcal{U}}$ is homotopy equivalent to M.

Quiz 3.2. What is a fiber of the map $B\Phi : B\Gamma_{\mathcal{U}} \longrightarrow M$? How is it contractible?

There is a Γ structure on $B\Gamma$. The simplex $\triangle^n$ has the barycentric subdivision $bs\triangle^n$. By gathering the simplices of $bs\triangle^n$ adjacent to one of the vertices of $\triangle^n$, we see that $\triangle^n$ is covered by $n+1$ cubes $\{C_k\}_{k=0,\dots,n}$. By enlarging these cubes a little, we have an open cover of $\bigsqcup \Gamma^{(n)} \times \triangle^n$. We define the maps

$$\begin{aligned} (\gamma_1,\dots,\gamma_n)\times C_k &\longmapsto t(\gamma_{k+1}) = s(\gamma_k) \quad \text{and} \\ (\gamma_1,\dots,\gamma_n)\times (C_j\cap C_k) &\longmapsto \gamma_{j+1}\dots\gamma_k \end{aligned}$$

for $j \le k$. Then these defines the Γ structure on $B\Gamma$.



Quiz 3.3. How are C_k? How is the pullback by $B\Gamma_{\mathcal{U}} \longrightarrow B\Gamma$?

Why is $B\Gamma$ called the classifying space? What does it classifies? Classifying means the homotopy class of maps to the space corresponds bijectively to an isomorphism class of something. For example the isomorphism class of real vector bundles are classified by the homotopy class of maps to $BO(n)$. For the Γ structures in general, the classifying space $B\Gamma$ only classifies the homotopy class of Γ structures. It looks an ultra tautological statement!

Anyway, we define the homotopy of Γ structures by declaring that the restrictions to $M\times\{0\}$ and $M\times\{1\}$ of a Γ structure on $M\times[0,1]$ are homotopic.

Then we have the following obvious theorem.

Theorem 3.4. *The set of homotopy classes of Γ structures on M is in bijective correspondence with $[M, B\Gamma]$.*

As is already explained, codimension q, C^r foliations are Γ_q^r structures. Hence they are classified by Haefliger's classifying space $B\Gamma_q^r$.

In fact, the classifying spaces have been very useful in classifying the foliations of open manifolds up to integrable homotopy by Haefliger and the foliations of closed manifolds up to concordance by Thurston.

4. Foliated microbundle

In order to say something about foliations, we need to associate a foliated space to a Γ_q^r structure. A Γ_q^r structure is given by the homomorphism $\Gamma_{\mathcal{U}} \longrightarrow \Gamma_q^r$, that is, $\{U_i, f_i, \gamma_{ij}^x\}_{i,j\in I}$ satisfying the cocycle condition $\gamma_{ij}^x\gamma_{jk}^x = \gamma_{ik}^x$. It is easier to think about a good finite cover $\mathcal{U}$ for a compact manifold M and take a finite set of representatives $\{\gamma_{ij}\}$. Notice that we are omitting ^{x}. Consider $\bigsqcup_i U_i \times \mathbf{R}^q/\sim$, where the identification is given by $((x \in U_i), y_i) \sim ((x \in U_j), y_j)$ if $y_i = \gamma_{ij}y_j$. There should be some inconvenience for a point $x \in U_i \cap U_j \cap U_k$, where $\gamma_{ij}\gamma_{jk} = \gamma_{ik}$ holds only in a neighborhood of $f_k(x)$. Hence we consider the graph $\{(x, f_i(x))\} \subset U_i \times \mathbf{R}^q$ of f_i and think of the space obtained from the union of neighborhoods of these graphs by the identification. This space (germinally well defined) is called foliated microbundle associated to the Γ_n^r structure. This space is an open manifold E with foliation induced from the horizontal foliations on $U_i \times \mathbf{R}^q$. It is important to notice that there are a projection $p : E \longrightarrow M$ given by the projections to the first factor and a map $s : M \longrightarrow E$ defined from the graphs of f_i. Thus we have another obvious theorem.

Theorem 4.1. *For a Γ_n^r structure $\mathcal{H}$ on M, there exists a foliated space $(E, \mathcal{F}_{\mathcal{H}})$ (a foliated microbundle) and maps* $\begin{array}{c} E \\ p\!\downarrow\uparrow\! s \\ M \end{array}$ *such that $p \circ s = \mathrm{id}_M$ and $s^*\mathcal{F} = \mathcal{H}$.*

Quiz 4.2. The normal bundle of the section $s(M)$ is defined by the homomorphism, $\Gamma_q^r \longrightarrow GL(q;\mathbf{R})$ which maps $\gamma(x) \xleftarrow{\gamma} x$ to $* \xleftarrow{D\gamma} *$.

Quiz 4.3. The topological groupoid G with only one object $*$ is the topological group. What is a *group* structure over a space M ?

Quiz 4.4. For a codimension q, C^r foliation $\mathcal{F}$ of M, let $\nu\mathcal{F}$ denote the normal bundle of the foliation. Taking a Riemannian metric on M, we have the exponential map $\mathrm{Exp} : \nu\mathcal{F} \longrightarrow M$. Then $(E, \mathcal{F}_{\mathcal{F}})$ is isomorphic to $(\nu\mathcal{F}, \mathrm{Exp}^*\mathcal{F})$.

5. Theorem of existence of foliations

Since the normal bundles of homotopic Γ_q^r structures are isomorphic, the normal bundle $\nu\mathcal{H}$ of a Γ_q^r structure $\mathcal{H}$ on M homotopic to a foliation is isomorphic to a subbundle of the tangent bundle TM of M.

The converse is the theorem of existence of foliations by Gromov-Phillips-Haefliger-Thurston.

Theorem 5.1. *For a Γ_q^r structure $\mathcal{H}$ on M, assume that the normal bundle $\nu\mathcal{H}$ is isomorphic to a subbundle of the tangent bundle of M. Then there exists a foliation $\mathcal{F}$ of M which is homotopic to $\mathcal{H}$ as Γ_q^r structure.*

The proof starts from the foliated microbundle $(E, \mathcal{F}_{\mathcal{H}})$. If one can deform the section s by homotopy in E so that the new section is transverse to $\mathcal{F}_{\mathcal{H}}$, then the theorem is proved. If it is not easy to do so, we modify $\mathcal{F}_{\mathcal{H}}$ and try to make s transverse again.

Quiz 5.2. What is the relative version of the above theorem? How the relative version of the theorem leads to the theorem of classification of foliations up to concordance?

6. Foliated bundles and foliated products and Mather-Thurston theory.

A rich source of foliations is the representation in transformation group, that is, foliated bundles and foliated products.

For a manifold M, $\mathrm{Diff}(M)$ denotes the group of diffeomorphisms of M. For a connected manifold N and a homomorphism $h : \pi_1(N) \longrightarrow \mathrm{Diff}(M)$, we can associate the space $E = \widetilde{N} \times M/\pi_1(N)$, where $\alpha \in \pi_1(N)$ acts on $\widetilde{N} \times M$ by the covering transformations and through the homomorphism h: $\alpha(\widetilde{u}, x) = (\alpha\widetilde{u}, h(\alpha)(x))$. Then E has the foliation $\mathcal{F}$ induced from the horizontal foliation of $\widetilde{N} \times M$. $(E, \mathcal{F})$ is a foliated M bundle over N. The homomorphism h is called the global holonomy.

Quiz 6.1. Foliations of an M bundle over N transverse to the fibers are foliated M bundles.

Quiz 6.2. This foliated M bundle structure can be viewed as a group structure on N, where the group is $\mathrm{Diff}(M)$ with the discrete topology, denoted by $\mathrm{Diff}(M)^\delta$. This structure has the classifying space $B\mathrm{Diff}(M)^\delta$.

Example 6.3. The Anosov foliations associated with the negatively curved closed manifolds are foliated bundles.

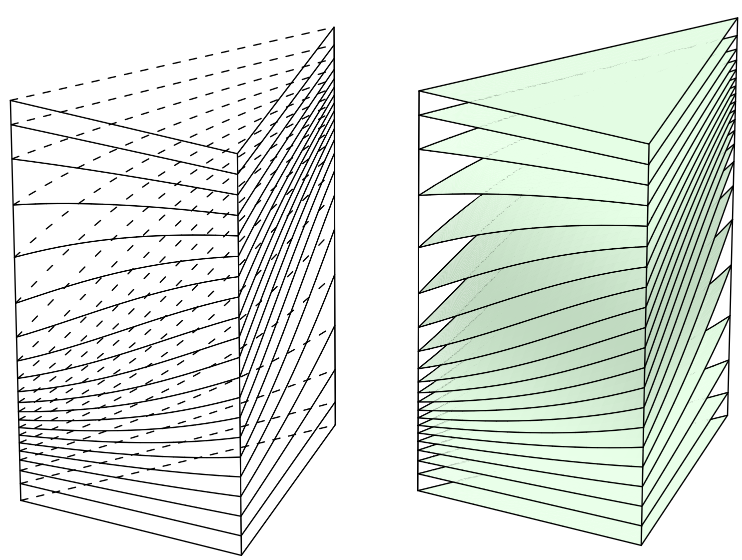
Quiz 6.4. For the group $\mathrm{Diff}^r(M)$ of C^r diffeomorphisms of M ($r \geq 1$), the composition $(f_1, f_2) \longmapsto f_1 f_2$ is smooth in the first factor. For a C^r foliated M bundle $(E, \mathcal{F})$ over a smooth manifold N, leaves of $\mathcal{F}$ are smooth submanifolds of E (by changing by a C^r isotopy). For a C^r foliation of a manifold N, thinking about the foliated microbundle and the pull back by the section s, leaves of the foliations are smooth submanifolds of the manifold N (by changing by a C^r isotopy).

Definition 6.5. A foliate product is a foliated bundle with the total space E identified with $N \times M$.

The classifying space for the foliated products is constructed explicitly.

Let us consider the set all foliated M product structures over the standard n-simplex $\triangle^n$, that is the foliations of $\triangle^n \times M$ transverse to the fibers of the first factor projection. Since the restrictions of a foliated M product structure over $\triangle^n$ to its faces are foliated M product structures over $\triangle^{n-1}$, there is a natural way to identify the disjoint union of all foliated M product structures over the standard n-simplex $\triangle^n$ for $n \in \mathbf{Z}_{\geq 0}$. Then we obtain a foliated M product over a certain space, which is the classifying space for them. The classifying space is denoted by $B\overline{\mathrm{Diff}}(M)$.

Quiz 6.6. There is a bijective correspondence between the set of all foliated M product structures over the standard n-simplex $\triangle^n$ and $\mathrm{Map}(\triangle^n, \mathrm{Diff}(M))/\mathrm{Diff}(M)$. $\sigma : \triangle^n \longrightarrow \mathrm{Diff}(M)$ determines the foliation on $\triangle^n \times M$ whose leaf passing through (t, x) is given by $\{(s, \sigma(s)(\sigma(t))^{-1}(x)) \ ; \ s \in \triangle^n\}$.



Quiz 6.7. For a foliated M product over the standard n-simplex $\triangle^n$, we can consider its subdivision in the direction of the base space $\triangle^n$. Observe that, after a fine subdivision, the restriction of the foliated product over one of the simplices is much closer to the horizontal foliation if we rescale the simplex as the standard simplex.

Quiz 6.8. Foliated product structure is homotopically a group structure. The group $\overline{G}$ is given as follows: an element of $\overline{G}$ is a diffeomorphism f_1 with the class of an isotopy $[f_t]$ from the identity. The composition is given by $[f_t][g_t] = [f_t g_t]$.

Recall that the foliated microbundles look like foliated bundles because the foliations are transverse to the fiber of projections. The difference is that the global holonomy is not defined. One can try to embed the foliated microbundle in a foliated bundle.

It can be done in some special cases (codimension 1 foliations without holonomy, analytic foliations with holonomy being the restrictions of global analytic diffeomorphisms, in particular, transversely geometric foliations). In the differentiable case, one can do in a partial way.

Theorem 6.9 (Mather-Thurston)**.**

$$H_*(B\overline{\mathrm{Diff}}{}^r_c(\mathbf{R}^n); \mathbf{Z}) \cong H_*(\Omega^n B\overline{\Gamma}{}^r_n; \mathbf{Z}).$$

Quiz 6.10. What is the map $B\overline{\mathrm{Diff}}{}^r_c(\mathbf{R}^n) \longrightarrow \Omega^n B\overline{\Gamma}{}^r_n$?

$B\overline{\Gamma}{}^r_n$ is the classifying space for the Γ^r_n structures with trivialized normal bundles. It is better to think that this space classifies the Γ^r_n structures over N with foliated microbundles embedded in $N \times \mathbf{R}^n$.

The group $\mathrm{Diff}^r_c(\mathbf{R}^n)$ is the group of diffeomorphisms of $\mathbf{R}^n$ with compact support. That is, its element is the identity outside a certain compact set. The space $B\overline{\mathrm{Diff}}{}^r_c(\mathbf{R}^n)$ is the union of all foliated $\mathbf{R}^n$ products with compact support over the simplices. They are perturbations of the horizontal foliation in compact subspaces.

A cheap way to understand Mather-Thurston's theorem for the codimension 1 foliations on 3-manifold N is as follows: The foliated microbundle is embedded in $N \times \mathbf{R}$ and the section s is to $\{0\} \times \mathbf{R}$. Take a sufficiently fine Heegaard splitting of N.

$$N = H_g \cup_{\Sigma_{2g}} H'_g.$$

If we look at the foliated microbundle restricted over H_g, its local holonomy germs near 0 can be prolonged as diffeomorphisms near $[-1, 0]$ identity near -1. We do this for the generators of $\pi_1(H_g)$. In the same way, the local holonomy germs near 0 of the foliated microbundle restricted over H'_g can be prolonged as diffeomorphisms near $[0, 1]$

identity near 1. Then, on the foliated microbundle restricted over Σ_{2g} is embedded in a foliated $[-1,1]$ product. Moreover the zero section s is homologous to $\Sigma_{2g} \times [-1,1]$ in the homology relative to the part with horizontal foliations $(H_g \times \{-1\} \cup H'_g \times \{1\})$.

Mather-Thurston's theorem says that when the foliated microbundle is embedded in $N \times \mathbf{R}^n$, we can deform the foliation so that the leaves are horizontal in a large part and the section s is homologous to the total space of foliated product relative to the part with horizontal foliations.

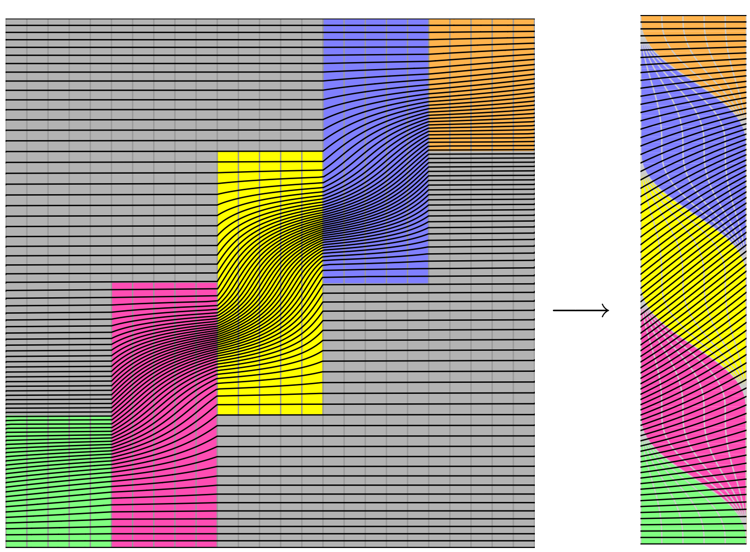
The key for introducing a large horizontal part is the following fragmentation.

First we look at fragmentation for foliated products. For a perturbation of the horizontal foliation of $\triangle^n \times M$, we can perform the following fragmentation operation.

Let $\{\mu_i\}_{i=1,\ldots,K}$ be a partition of unity for M. Put $\nu_j = \sum_{i=1}^{j} \mu_i$. Put $K \cdot \triangle^n = \{(u_1,\ldots,u_n) \in \mathbf{R}^n \ ; \ K \geq u_1 \geq \cdots \geq u_n \geq 0\}$ and $\triangle^n = 1 \cdot \triangle^n$. Define a map $K \cdot \triangle^n \times M \longrightarrow \triangle^n \times M$ by

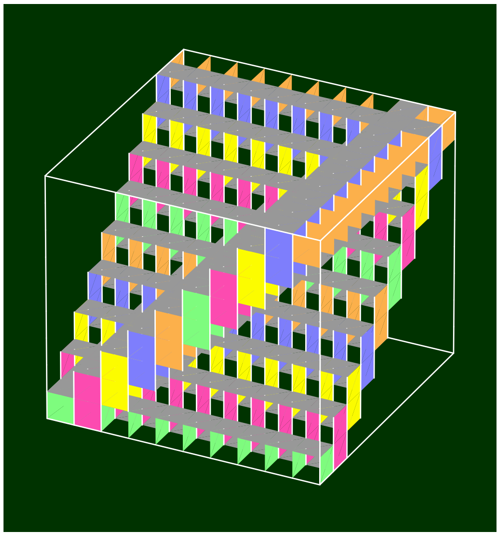
$$((u_1,\ldots,u_n),x) \longmapsto ((v_1,\ldots,v_n),x),$$

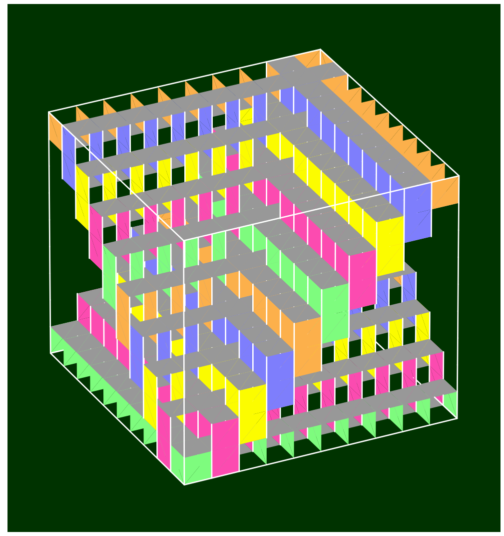
where $v_i = \nu_{[u_i]}(x) + (\nu_{[u_i]+1}(x) - \nu_{[u_i]}(x))(u_i - [u_i])$. This map is level preserving and if a foliation $\mathcal{F}$ is sufficiently close to the horizontal foliation this map is transverse to the foliation. Hence the pull back is a foliation.



For a foliated M-product over N, we take a triangulation by an ordered complex and apply the fragmentation to each ordered simplex simultaneously. Note that the fragmentation commutes with the face maps in ordered simplices.

Quiz 6.11. How is the support of the pullback foliation over $[i_1 - 1, i_1] \times \cdots \times [i_n - 1, i_n]$?





The fragmentation is homotopic to the map

$$(1/K) \times \mathrm{id} : ((u_1, \ldots, u_n), x) \longmapsto ((u_1/K, \ldots, u_n/K), x),$$

for which the pull back just changes the parametrization in the base direction. In other words, the fragmented foliated product structure over N is homotopic to the original foliated product structure.

The fragmentation homotopy can be applied to foliated microbundles which are embedded in $N \times \mathbf{R}^n$. We are assuming that N is triangulated as an ordered simplicial complex. Then we have fragmented foliated microbundle over N. We see that most of the part of $N \times \mathbf{R}^n$ is foliated horizontally.

However we need a lot more argument to say that the section s is homologous to the total space of a foliated $\mathbf{R}^n$ product with compact support.

Now consider how one can do with the section $s : N \longrightarrow N \times \mathbf{R}^n$. On the dual cell decomposition, over the cells of dimension $\dim N$ dual to the vertices the section s can be deformed by homotopy to horizontal sections to the part foliated horizontally. Over the cells dual to the edges, the sections defined over the cells dual to the vertices can be joined by edges along the fiber. Then we look at the part over the cells dual to the 2-simplices and find 2-simplices along the fiber filling the defined edges along the fiber, ... One can find $(n-1)$-simplices along the fiber over the cells dual to $(\dim N - n + 1)$-simplices which are in the part foliated horizontally. We need to understand the situation over the cells dual to $(\dim N - n)$-simplices. There we find foliated

$\mathbf{R}^n$ products with support in n-simplices. This is only the beginning of the argument. We need to look at the situation over the cells dual to $(\dim N - n - 1)$-simplices, and we find that foliated $\mathbf{R}^n$ products with different supports are attached along the fiber, and then over the cells dual to $(\dim N - n - 2)$-simplices, the life is more complicated, ... This argument would help a little to understand why the proof in [7] begins with looking at attaching foliated products with different supports along the fiber.

We have some hope to understand the classifying space for other transitive groupoids in a similar way.

For $B\overline{\Gamma}_n^{\mathrm{vol}}$, there are a pile of works by McDuff, Hurder, ... For $B\overline{\Gamma}_{2n}^{\mathrm{symplect}}$, see the works by Banyaga for a few known facts. Almost nothing is known for $B\overline{\Gamma}_{2n-1}^{\mathrm{contact}}$.

7. Topology of the classifying spaces $B\Gamma$

The topology of classifying spaces $B\Gamma$ or $B\overline{\Gamma}$ is now important. The cohomology classes of $B\Gamma_q^r$ or $B\overline{\Gamma}_q^r$ are the characteristic classes of foliations and they give rise to cobordism invariant. We give a partial list of known results. The results for $B\overline{\Gamma}_n^r$ $(r \leq \infty)$ is obtained by using Mather-Thurston's theorem. So if the universal covering group $\widetilde{\mathrm{Diff}}_c^r(\mathbf{R}^n)_0$ of the connected component of the identity of $\mathrm{Diff}_c^r(\mathbf{R}^n)$ is a perfect group, $H_1(B\overline{\mathrm{Diff}}_c^r(\mathbf{R}^n); \mathbf{Z}) = 0$ and $B\overline{\Gamma}_n^r$ is $n+1$-connected.

- $B\overline{\Gamma}_1^\omega$ is a $K(\pi, 1)$-space (Haefliger),
  $H_1(B\overline{\Gamma}_1^\omega; \mathbf{Z}) = 0$,
  $H_2(B\overline{\Gamma}_1^\omega; \mathbf{Z}) =?$,
  $H_3(B\overline{\Gamma}_1^\omega; \mathbf{Z}) \geq \mathbf{R}$ (Godbillon-Vey, Thurston), ...
- $B\overline{\Gamma}_n^{\mathbf{C}}$ is $n+1$-connected (Adachi).
- $B\overline{\Gamma}_n^r$ is n-connected.
- $B\overline{\Gamma}_n^r$ is $n+1$-connected $(r \neq n+1)$ (Mather).
- $B\overline{\Gamma}_n^0$, $B\overline{\Gamma}_n^L$, $B\overline{\Gamma}_n^1$ is contractible.
- The connectivity of $B\overline{\Gamma}_n^r$ increases as $r \searrow 1$.
- $\pi_{2n+1}(B\overline{\Gamma}_n^r) \geq \mathbf{R}$ $(r > 2 - 1/(n+1))$. (The theory of characteristic classes for the foliations by Gelfand-Fuks, ...)
- The cohomology of $B\overline{\Gamma}_2^{\mathrm{area}}$ is studied by Gelfand-Kalinin-Fuks, Metoki, ...
- The cohomology of $B\overline{\Gamma}_n^{\mathbf{C}}$ is studied by Asuke.
- $B\overline{\Gamma}_n^{PL}$
  $B\overline{\Gamma}_1^{PL} \simeq B\mathbf{R}^\delta * B\mathbf{R}^\delta$ (Greenberg).
  $B\overline{\Gamma}_2^{PL\mathrm{vol}}$??

8. Foliations and groupoid

Recall that for a codimension q foliation $\mathcal{F}$ on a closed manifold M, we have the homomorphism $\Phi : \Gamma_{\mathcal{U}} \longrightarrow \Gamma_q^r$. We can look at the subgroupoid of Γ_q^r generated by the image of this homomorphism. These should be finitely generated or compactly generated groupoids. The notion of finitely generated or compactly generated groupoids is studied by Haefliger, Ghys, Meigniez and others.

Problem by Haefliger: Is a finitely generated groupoid with compact section a groupoid associated with a foliation on a compact manifold ?

- The action of a compact group or a nilpotent group on itself.
- Lie foliations (Ghys).
- Solvable groups, eg. Aff(1) (Meigniez).
- $SL(2;\mathbf{R})$??

There is a characterization of the classifying space for Γ structure.

Theorem 8.1 (Segal-Haefliger). *Let $(X, \mathcal{F})$ be a foliation on a compact manifold. Let $\Gamma_{\mathcal{F}}$ denote the groupoid associated with $\mathcal{F}$. Then X is homotopy equivalent to the classifying space $B\Gamma_{\mathcal{F}}$ for the groupoid $\Gamma_{\mathcal{F}}$ if and only if the holonomy covering of any leaf is contractible.*

The holonomy covering $\widetilde{L}$ of a leaf L of $\mathcal{F}$ is the covering space corresponding to the kernel of the holonomy homomorphism $\pi_1(L) \longrightarrow \mathrm{Diff}(\mathbf{R}^q, 0)$, where $\mathrm{Diff}(\mathbf{R}^q, 0)$ denote the group of germs of local diffeomorphism of neighborhoods of 0.

We have the following constructions.
Foliation $\mathcal{F}$ on $M \rightsquigarrow$ Groupoid $\Gamma_{\mathcal{F}} \rightsquigarrow$ Classifying space $B\Gamma_{\mathcal{F}}$
Now the question is the following.

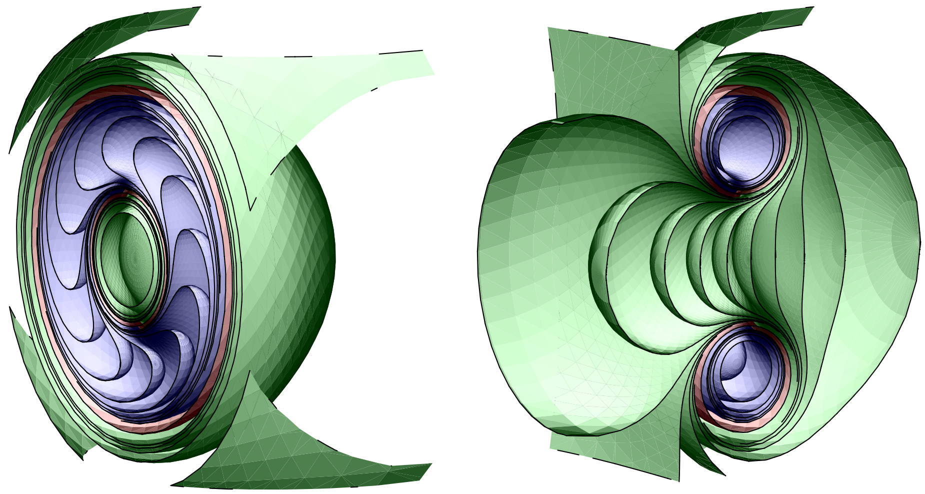
Problem. Is $B\mathcal{F} : M \longrightarrow B\Gamma_{\mathcal{F}}$ a homotopy equivalence?

Definition 8.2. A typical foliation is a foliation which is the classifying space for itself.

There are works by Salem, Gusmão, ... One can also ask whether $B\mathcal{F}$ induces the isomorphism in π_1, ...

Example 8.3.

- Flows are generically typical.
- The Reeb foliations on S^3 are typical.
- Foliation by planes on T^3 are typical.
- Anosov foliations are typical.



Quiz 8.4. Which 3-manifold admits typical codimension 1 foliations?

Quiz 8.5. Depth 1 foliations are usually not typical. Which are the typical foliations of depth 1?

Proposition 8.6. *There exist typical codimension 1 foliations on graph manifolds.*

Quiz 8.7. How about typical foliations with singularity?

References

1. A. Banyaga, The structure of classical diffeomorphism groups. Mathematics and its Applications, 400. Kluwer Academic Publishers Group, Dordrecht, 1997.
2. R. Bott, *Raoul Bott: collected papers*. Vol. 3. Foliations. Edited by Robert D. MacPherson. Contemporary Mathematicians. Birkhäuser Boston, Inc., Boston, MA, 1995.
3. A. Haefliger, *Structure feuilletées et cohomologie à valeur dans un faisceau de groupoïdes*, Comment. Math. Helvet., 32 (1958), 248–329.
4. A. Haefliger, *Differential cohomology*, Differential topology (Varenna, 1976), pp. 19–70, Liguori, Naples, 1979.
5. A. Haefliger, *Groupoïdes d'holonomie et classifiants*, Structure transverse des feulletages (Toulouse, 1982). Astérisque No. 116 (1984), 70–97.
6. H. B. Lawson Jr, *The quantitative theory of foliations*, Expository lectures from the CBMS Regional Conference held at Washington University, St. Louis, Mo., January 6–10, 1975. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 27. American Mathematical Society, Providence, R. I., 1977.
7. J. N. Mather, *On the homology of Haefliger's classifying space*, Differential topology (Varenna, 1976), pp. 71–116, Liguori, Naples, 1979.