A List of Questions about Foliations

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Contents

The following collection of questions in the theory of foliations are a sampling of the many open problems in the field. They where collected mainly from the participants to the meeting on foliation held at P.U.C. (Rio de Janeiro) in January 1992. One hopes that progress on any of these will lead to new insights into foliation theory.

A Examples and counter-examples of foliations

A.1. Elmar Vogt

A.1.1 Question : *Let $\mathcal{F}$ be a compact foliation (i.e. a foliation with all leaves compact) of codimension 3 on a compact manifold. Let $\mathcal{B}$ be the first bad set of the Epstein hierarchy of $\mathcal{F}$, i.e. $\mathcal{B}$ is the union of all leaves of $\mathcal{F}$ with infinite holonomy group. If $\mathcal{B}$ is not empty, can $\mathcal{B}$ be a manifold ?*

Remarks : In [44], arguments (due to M. Hirsch and D.B.A. Epstein) are presented that make it plausible that $\mathcal{B}$ cannot be a manifold , at least in the case of a 1-dimensional foliation. That there are compact foliations of codimension 3 on compact manifolds with non-empty $\mathcal{B}$ was established by D.B.A. Epstein and E. Vogt [13].

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A.1.2 Question : *Let $\mathcal{F}$ be a compact foliation on a compact manifold. Assume that the first bad set, $\mathcal{B}$, of $\mathcal{F}$ is not empty.*
Is there any restriction on the topological type of the typical leaves of $\mathcal{F}$?
In particular, is it possible to have simply connected leaves in such a foliation ?

Remarks : A leaf of compact foliation is called typical if its holonomy group is trivial. The typical leaves of a compact foliation form an open dense set.

- Any closed manifold can occur as a typical leaf of a compact foliation of any given codimension ≥ 2 with non-empty $\mathcal{B}$ on a *non-compact* manifold (see E. Vogt [54]).

(A.1.3) We use the notation established above. In all known examples of compact foliations with non-empty bad set each leaf in the bad set has a finite cover with infinite first homology group.

A.1.3 Question : *Is this true in general ?*
Remarks : The answer is yes for compact foliations of codimension ≤ 3 and for compact foliations of codimension ≤ 4 on compact manifolds (see E. Vogt [54]).

(A.1.4) It is possible to foliate $\mathbf{R}^n$ by tori of any codimension≥ 2 (E. Vogt, Preprint 1991). Can the leaves of such a foliation be the orbits of an $\mathbf{R}^k$-action ? Or more generally :

A.1.4. Question : *For which q does there exist a locally free $\mathbf{R}^k$-action on $\mathbf{R}^{k+q}$ with all orbits compact ?*
Remarks : Using products of circle foliations of $\mathbf{R}^3$ there exist such actions for $q \geq 2k$. The question is to find the smallest q.

(A.1.5) What can be said about the leaves of compact foliations of Euclidean spaces ? More specifically, considering the construction of such foliations mentioned in the preceding problem, the following questions look interesting :

A.1.5. Questions : *Is every leaf in the bad set covered by a torus ?*
Can a closed hyperbolic 3-manifold be a leaf of a compact foliation of $\mathbf{R}^5$ or, for that matter, of any Euclidean space ?
Can a closed hyperbolic 3-manifold appear in the bad set of a compact foliation of a Euclidean space ?

(A.1.6) The most intriguing question for compact foliations on Euclidean spaces is the question of smoothness of such foliations. Specifically

A.1.6. Questions : *Does there exist a C^1-foliation of $\mathbf{R}^3$ by circles ?*
(Possibly easier) Does there exist a C^1-foliation of some $\mathbf{R}^n$ by circles ?
(D.B.A. Epstein) What about real analytic examples ?

A.2. Paul Schweitzer

A.2.1 Question : *Does every $\mathcal{C}^2$ (or $\mathcal{C}^\infty$) codimension one foliation of S^5 or $S^1 \times S^4$ have a compact leaf ?*
Remark : The following result is known : on every closed smooth manifold $M^n, n \geq 4$, with Euler characteristic zero there exists a $\mathcal{C}^1$ codimension one foliation with no compact leaf (P. Schweitzer, [43]).

A.3. Steve Hurder

We assume that V is a compact manifold without boundary, and $\mathcal{F}$ is a codimension-q foliation with transverse differentiability class C^r. A foliation is *almost compact* if there are at most a finite number of non-compact leaves.

A.3.1. Problem. *For $0 \leq r \leq \infty$ does there exists a C^r-foliation $\mathcal{F}$ of V with codimension $q \geq 1$ such that $\mathcal{F}$ has exactly one non-compact leaf ? Is is possible to find such an $\mathcal{F}$ with finitely many non-compact leaves? with countably many non-compact leaves?*

The motivation for this question is the observation

Theorem (Theorem 2, [25])*Let L be a non-proper leaf of a topological (C^0) foliation $\mathcal{F}$. Then the topological closure $\bar{L}$ contains uncountably many non-compact leaves.*

Hence, if there is a non-proper leaf for a topological foliation $\mathcal{F}$, then the set of non-compact leaves must be uncountable. But it is not known whether there exists proper foliations of positive codimension with a countable, non-empty set of non-compact leaves. A natural starting point for this problem is to investigate whether the techniques developed by Elmar Vogt for constructing compact foliations on open manifolds [53, 55] can be used to produce foliations of compact manifolds with only exceptional non-compact leaves.

A.3.2. Problem. *Let $\mathcal{F}$ a C^1-foliation of V with codimension $q \geq 1$. Show that either all leaves of $\mathcal{F}$ are compact, or the set of non-compact leaves for $\mathcal{F}$ has positive Lebesgue measure.*

Remarks : Here is an example that shows the set of non-compact leaves need not have positive Lebesgue measure for a topological foliation, even though there are uncountably many non-compact leaves.

Example 2. Construct the "Cantor function" $f : [0,1] \to [0,1]$ which is continuous, has range in the dyadic rationals on the complement of the "middle third" Cantor set $\mathbf{K} \subset [0,1]$, and is increasing on the complement. The range of f is all of $[0,1]$, and the image of the Cantor set $\mathbf{K}$ (which has Lebesgue measure zero) is a set of full measure 1. In particular, the Cantor function f is not absolutely continuous.

Define an "exotic" action ϕ of $\mathbf{Z}$ on the cylinder $[0,2] \times \mathbf{S}^1$ using the Cantor function to define a homeomorphism : For $[r,\theta] \in [0,2] \times \mathbf{S}^1$, set

$$\phi[r,\theta] = \left\{ \begin{array}{cl} [r, \theta + 2\pi f(r)] & \text{for } \ 0 \leq r \leq 1 \\ [r, \theta + 2\pi f(2-r)] & \text{for } \ 1 \leq r \leq 2 \end{array} \right\}$$

Clearly, the set of periodic orbits for ϕ has full measure, and the set of non-periodic orbits is non-empty. The circles $\{0\} \times \mathbf{S}^1$ and $\{2\} \times \mathbf{S}^1$ are fixed by the action, so we can identify them to points to obtain a homeomorphism $\hat{\phi} : \mathbf{S}^2 \to \mathbf{S}^2$. Suspend this action to obtain a 1-dimensional foliation of the manifold :

$$\mathbf{S}^1 \times \mathbf{S}^2 \simeq (\mathbf{R} \times \mathbf{S}^2)/(x, [r,\theta]) \sim (x+1, \hat{\phi}[r,\theta])$$

where almost every leaf is a circle, and which is a counter-example to Problem 2 in the topological category.

B Exceptional minimal sets

B.1. L. Conlon

Let $\mathcal{F}$ be a codimension one, C^2 foliation of a closed n-manifold M. Let $X \subset M$ be an exceptional minimal set of $\mathcal{F}$. The following are moderately old problems.

B.1.1. Question : *(Dippolito) Let* L *be a semiproper leaf of* $X, x \in X$*, and let* $H_x(L, X)$ *be the germinal homology group of* L *at* x *relative to* X*.*

Is $H_x(L, X)$ *infinite cyclic ?*

Remarks : Hector proved in his thesis that the infinite jet of holonomy is infinite cyclic.

B.1.2. Question : *(Hector) Must* $M \setminus X$ *have only finitely many components ? (This is known to be false for* C^1 *foliations).*

B.1.3. Question : *(?) Must the Lebesgue measure* $\mid X \mid= 0$?

Remarks : This is important for the study of $GV(\mathcal{F})$. It would show that the only resilient leaves that could force $GV(\mathcal{F}) \neq 0$ would be locally dense ones.

-These questions have answer "yes" for Markov minimal sets Λ. i.e, the holonomy pseudogroup $\Gamma \mid X$ is generated by a (1-sided) subshift of finite type. These examples, or minor variations of them, are the typical examples of exceptional minimal sets.

B.1.4. Problem. *Can* X *be approximated by a sequence* $\{X_x\}$ *of Markov minimal laminations in such a way that* $\mid X \mid= \lim_{n\to\infty} \mid X_n \mid= 0$? *Could the other questions be resolved by such a technique* ?

B.2. Rémi Langevin (after C. Camacho)

B.2.1. Question : *Holomorphic foliations of* $CP(2)$ *need to have singularities. Does any leaf of an holomorphic foliation* $\mathcal{F}$ *of* $CP(2)$ *contain a singular point in its closure* ?

B.2.2. Question : *In other words, does there exists a holomorphic foliation of* $CP(2)$ *admitting an exceptional minimal set ? (the case of a compact leaf has been ruled out).*

Remarks. Camacho, Lins and Sad [9] proved that a holomorphic foliation of $CP(2)$ admits at most one exceptional minimal set.

- Bonatti, Langevin and Moussu [6] proved that, if such a minimal set exists, at least one leaf of it contains a loop with contracting linear holonomy.

C Groups of diffeomorphisms or germs

C.1. Etienne Ghys

C.1.1. Question : *Let* **eu** *be the Euler class in* $H^2(\mathrm{Diff}^\omega_+(S^1))$*, where* $\mathrm{Diff}^\omega_+(S^1)$ *means orientation preserving, real analytic diffeomorphisms of the circle with discrete topology. Are all powers* $\mathbf{eu}^k \in H^{2k}(Diff^\omega_+(S^1))$ non-zero for all $(k \geq 2)$.

Remark : For $\mathcal{C}^\infty$-diffeomorphisms, the powers of "**eu**" were shown to be non-zero by S. Morita [39].

C.2. Takashi Tsuboi

The *Zygmund class* of functions was originally introduced in the study of singular integral operators (cf. [35]), but remarkably this differentiability class is also proving to be central for the study of 1-dimensional dynamical systems (cf. [29, 45, 14]). Let $\mathrm{Diff}^{1+Z}_+(S^1)$ denote the group of orientation-preserving diffeomorphisms of the circle whose first derivative is in the "big Zygmund" class – this is the maximal class of maps for which the "Denjoy construction" can be performed. Let $\mathrm{Diff}^{1+z}_+(S^1)$ denote the group of orientation-preserving diffeomorphisms of the circle whose first derivative is in the "little Zygmund" class – this group has many of the properties of the group of C^2-diffeomorphisms.

C.2.1. Question :

a) is the group $\mathrm{Diff}^{1+Z}_+(S^1)$ *perfect ?*

b) is the group $\mathrm{Diff}^{1+z}_+(S^1)$ *perfect ?*

C.2.2. Question : *Let* $\rho : \mathbf{Z}^N \to \mathrm{Diff}^{1+\alpha}_C(\mathbf{R})$ *be a homeomorphism such that there exists an open interval* $U \subset \mathbf{R}$ *with* $\rho(a)(U) \cap \rho(a')(U) = \phi$ *for* a and a' *two different points of* $\mathbf{Z}^N$*. Show that* $\alpha < 1/(N-1)$*. It is true for* $N = 2$.

C.3. David Tischler

Let M be a manifold with a distinguished point x_0. The map $\Pi : \mathrm{Diff}(M) \to M$ is defined by $\Pi(\phi) = \phi(x_0)$.

C.3.1. Question : *When is this a trivial principal* $\mathrm{Diff}_{x_0}(M)$ *bundle ?*

Remark : (S. Hurder). The answer is yes when $M = G$, a compact Lie group; but no when $M = G/K$, with G and K compact Lie groups such that M has non-zero Euler characteristic. The problem is equivalent to asking when does $\mathrm{Diff}(M)$ behave as if M were a Lie group. For example, is it true if M has the homotopy type of a Lie group? Is it true if M is Hopf?

C.4. José Luis Arraut and Nathan dos Santos

C.4.1. Problem. *Let* $\varphi : \mathbf{Z} \to Diff^\infty(S^1)$ *be a minimal action, i.e., the orbit* $O(z) = \{\varphi(n)(z), n \in \mathbf{Z}\}$ *of every point* $z \in S^1$ *is dense. Denote by* $C^\infty_\varphi(S')$ *the* $\mathbf{Z}$*-module of all* $\mathcal{C}^\infty$ *functions* $f : S^1 \to \mathbf{R}$ *with the multiplication rule* $n \cdot h = h \circ \varphi_n$*. Compute the cohomology* $H^1(\mathbf{Z}, C^\infty_\varphi(S'))$.

D Foliations from the dynamical view point

D.1. Christian Bonatti, Rémi Langevin et Claudio Possani.

There is a natural way to measure how the leaf L_x through a point x goes away from the leaf L_y through a nearby point y. Given $R > 0$, consider a path γ_x in L_x with origin x and length less or equal to R and project it locally on L_y starting at the point x (see [37]). Let $p_{\text{loc}}(\gamma)$ be the resulting path on L_y. We can perform the same construction with a path starting in y and projecting it onto L_x. Then we define

$$d_1 = \sup_{\gamma_x | l(\gamma_x) \leq R} \sup_t (d\gamma_x(t), p_{\text{loc}}\gamma_x(t)),$$

$$d_2 = \sup_{\gamma_y | l(\gamma_y) \leq R} \sup_t (d\gamma_y(t), p_{\text{loc}}\gamma_y(t)),$$

$$d_R(x,y) = \max(d_1, d_2).$$

Definition. A foliation $\mathcal{F}$ of a riemannian manifold (M, g) is expansive if there exists $\varepsilon > 0$ such that for each pair of points x and y in M, close enough to allow the above construction, there exists $R > 0$ such that $d_R(x,y) \geq \varepsilon$.

Inaba and Tsuchiya [32] proved that codimension 1 expansive foliations of a compact manifold have a resilient leaf. Adapting their method one can prove that the resilient leaves are in fact dense in that case and, if the foliation is of class $\mathcal{C}^2$, that a resilient leaf has some non trivial linear holonomy.

Remark : Hurder [24] proved that a codimension one foliation of class $C^{1+\alpha}$ with a resilient leaf, has a resilient leaf with contracting linear holonomy.

One can strengthen the definition of expansivity imposing some exponential rate of dilatation.

Definition. A foliation $\mathcal{F}$ is *weakly hyperbolic* if there exists $a > 0, \varepsilon > 0, \varepsilon_1 > 0$, such that if the $d(x,y) < \varepsilon_1$ one has :

$$d_R(x,y) \geq \varepsilon \quad \text{for} \quad R \geq a \cdot \text{Log}\ (\varepsilon / d_0(x,y))$$

where $d_0(x,y)$ is the transverse distance between x and y, obtained by taking $R = 0$ in the definition of d_R.

Using the Riemannian metric of M one can also define locally the direction and the strength of the infinitesimal contraction of the leaves, see [6], obtaining a vector field tangent to $\mathcal{F}$.

Definition. If the vector field of infinitesimal contraction has no sink we say that the foliation is strongly hyperbolic.

D.1.1. Question : *If the foliation is weakly hyperbolic for some Riemannian metric, is it true that there exists another Riemannian metric such that the foliation is strongly hyperbolic ?*

D.1.2. Question : *Is it reasonable to hope that strongly hyperbolic foliations have a Markov partition ?*

Modifying slightly the definition of expansivity, we can define foliations with "sensitive dependence on initial data", asking that for every x and every neighbourhood $v(x)$ of x there exists $y \in v(x)$ and $R(y)$ such that $d_{R(y)}(x, y) \geq \varepsilon$.

D.1.3. Question : *Do there exists nice examples of foliations with "sensitive dependence on initial data" which are not hyperbolic ?*

D.2. S. Matsumoto

Consider an Anosov flow $\{\varphi_t\}$ on a closed 3-manifold M^3. Under some smoothability assumption, the stable foliation is given by a 1-form ω such that $d\omega$ never vanishes. Now forget the flow $\{\varphi_t\}$ and simply consider a foliation $\mathcal{F}$ given by a 1-form ω such that $d\omega = \omega \wedge \eta$ never vanishes. Then we get a dim 1 foliation ("generalized horocycle foliation") $\mathcal{H}$ by setting $\omega = \eta = 0$. We can show the following :

(1) $\mathcal{F}$ is a foliation by planes and cylinders

(2) $\mathcal{H}$ is all-leaves-dense, unless M^3 is a T^2-bundle over S^1.

D.2.1. Question : *Construct a new example of such $\mathcal{F}$.*

D.2.2. Question : *Is $\mathcal{H}$ uniquely ergodic ?*

E Stability problems

E.1. Christian Bonatti

In [5] we prove :

Theorem. *Let M be a compact orientable 3-manifold, and let $\mathrm{Fol}^\infty(M)$ denote the space of $\mathcal{C}^\infty$ codimension 1, transversally oriented foliations on M. Then the set of foliation with no compact leaf of genus > 1 contains a dense open subset of $\mathrm{Fol}^\infty(M)$ for the $\mathcal{C}^0$ topology.*

To prove this result, we show that a compact leaf of genus ≥ 2 is always unstable (i.e. is not persistent) by $\mathcal{C}^0$ perturbations of the foliation. We have not succeeded in constructing a $\mathcal{C}^1$ perturbation of a foliation which modify non trivially the local dynamic of such a compact leaf.

E.1.1. Question : *Can a compact leaf of genus ≥ 2 be persistent by $\mathcal{C}^r$-small ($r \geq 1$) perturbations of the foliation ?*

A foliation $\mathcal{F}$ is *semi-stable* if every foliation $\mathcal{G}$ $\mathcal{C}^r$-close to $\mathcal{F}$ is semi-conjugated to $\mathcal{F}$ by a continuous map (which is not necessarily injective.)

E.1.2. Question : *Can a compact surface of genus ≥ 2 be a leaf of a $\mathcal{C}^r$-structurally stable (or semi-stable) foliation ?*

Remark : It should be possible to "open handles" of non compact leaves by $\mathcal{C}^0$-perturbations.

E.1.3. Question : *Denote by G_n the set of foliations $\mathcal{F} \in \mathrm{Fol}^\infty(M)$ such that all leaves of $\mathcal{F}$ have genus smaller than n. Does G_n (or $\bigcup_1^\infty G_n$) contain a $\mathcal{G}_\delta$ dense subset of $\mathrm{Fol}^\infty(M)$ for the $\mathcal{C}^r$-topology ?*

Remarks : Rosenberg and Roussarie [42] prove the non-existence of a $\mathcal{C}^1$-structurally stable foliation of S^3. The proof is that any foliation $\mathcal{F} \in \mathrm{Fol}(S^3)$ has a $\mathcal{C}^\infty$-flat compact leaf (topologically a torus) ; then they can"thicken" it, providing the perturbation. But the resulting foliation is only semi-conjugated to $\mathcal{F}$.

The Reeb foliation is not stable, but it is $\mathcal{C}^1$-structurally semi-stable.

F Classifying spaces and characteristic classes

F.1. Etienne Ghys

F.1.1. Question : *Let M^4 be a compact orientable manifold and*

$$R : \Pi_1(M^4) \to \mathrm{Homeo}^+(S^3)$$

be a representation.

Is it true that the Euler number of the associated S^3-bundle over M^4 is bounded by a number depending only on M^4? Find examples where this number is non zero.

Remark : The Milnor-Wood inequality [56] inequality gives such a bound for a 2-dimensional base M and fiber S^1. If the action of $\Pi_1(M^4)$ on S^3 is induced by a linear representation, then the answer is positive, using the bounded cohomology theory of Gromov[19]. One approach to the question is to establish that the generalized Euler class $\mathbf{eu_3} \in H^3(\mathrm{Homeo}^+(S^3); \mathbf{Z})$ is a bounded cohomology class.

F.2. Elmar Vogt

Homotopy type of $B\Gamma$

One model for $B\bar{\Gamma}_n$, the classifying space of codimension n Haefliger structures with trivialized normal bundle, is the total space of the $S0(n)$-bundle associated to the normal bundle map

$$\nu : B\Gamma_n^+ \to BS0(n),$$

where $B\Gamma_n^+$ classifies transversely oriented Haefliger structures of codimension n. Let $i : S0(n) \to B\bar{\Gamma}_n$ denote the inclusion of the fibre over the base point of $B\Gamma_n^+$.

F.2.1. Question : *Is the map i homotopic to* 0 ?

Remarks and further questions : The map i defines a Haefliger structure with trivialized normal bundle on $S0(n)$. Using the descriptions above, the Haefliger structure is simply the product structure on $S0(n) \times \mathbf{R}^n$ but with the twisted trivialization given by the natural action of $S0(n)$ on $\mathbf{R}^n$. Thus the question is equivalent to showing that this Haefliger structure is trivial as a *framed* Haefliger structure.

It is known that the answer to the question is "yes", if $n \leq 3$, for dimensional reasons because $B\bar{\Gamma}_n$ is $(n+1)$ - connected. The answer for $n = 4$ is also "yes" for dimensional reasons. This is due to the fact (a slip in nature) that a $S0(4)$

is diffeomorphic to $S^3 \times S0(3)$ with the diffeomorphism given by multiplication. (See Hurder [27]).

F.2.2. Problem : *Give a nice geometric null-concordance of the above framed Haefliger structures in these cases (i.e. for $n = 3$ or 4 – the case $n = 2$ is relatively easy).*

A positive answer to the question implies that the fibration over $S0(n)$ induced from the path fibration over $B\bar{\Gamma}_n$ by i is trivial, i.e. that

$$\Omega B\Gamma_n \simeq S0(n) \times \Omega B\bar{\Gamma}_n.$$

Steven Hurder conjectures in the paper cited above that this is true. This will imply in particular that the normal bundle map $\nu : B\Gamma_n^+ \to BS0(n)$ induces a split surjective map in homotopy.

The question above is probably very difficult. Here is a possibly easier question.

F.2.3. Question : *Let k be a field of characteristic 0, and let $i : S0(n) \to B\bar{\Gamma}_n$ be the map described above. Is the induced map in reduced homology with coefficients in k the 0-map ?*

Remark : For $k = \mathbf{Z}/p$ the answer to the corresponding question is "yes" since $H^*(BS0(n); \mathbf{Z}/p) \to H^*(B\Gamma_n^+; \mathbf{Z}/p)$ is injective by a theorem of Bott and Heitsch [7]

A theorem of M. Unsöld [52] assert that a positive answer to this question for a field k with characteristic $\neq 2$ will imply that

$$H_*(\Omega B\bar{\Gamma}_n; k) \to H_*(\Omega B\Gamma_n^+; k) \to H_*(S0(n); k)$$

is a short exact sequence of Hopf algebras. In particular, taking $k = \mathbf{Q}$, in the category of topological spaces

$$\Omega B\Gamma_n \simeq S0(n) \times B\bar{\Gamma}_n$$

would be true rationally (but this cannot happen as Hopf spaces).

Since the map induced by i in homology is multiplicative, it suffices to check that i_* vanishes on generators of the Hopf algebra $H_*(S0(n); k)$. Thus again for dimensional reasons the answer to the last question is "yes" for $n = 6$. But the smallest dimension where the answer is unknown is $n = 5$. Here one would have to answer the following

F.2.4. Question : *Does the codimension 5 framed Haefliger structure induced on S^7 from the above mentioned framed Haefliger structure on $SO(5)$ by the generator of $\pi_7(SO(5))$ bound homologically as a framed Haefliger structure?*

Because of the known connectivity of $B\bar{\Gamma}_5$ this is equivalent, up to torsion not recognized by k, to showing that this framed Haefliger structure on S^7 is trivial. This in turn implies directly that at least rationally the map induced by i in homotopy is surjective for $n = 5$.

Equivalent to F.2.1 (see J. Petro [40] and Hurder's question F.5.4) is the following often posed

F.2.5. Question: *Is* $H^k(BS0(n);\mathbf{Q}) \to H^k(B\Gamma_n^+;Q)$ *injective for* $k \leq 2n$, i.e. *injective up to the appearance of the Bott vanishing phenomenon ?*

Here the first non-trivial case is $n = 4, k = 8$. If p_1, p_2 are the Pontrjagin classes, it is a consequence of results of Thurston that neither p_1^2 nor p_2 get mapped to 0. Since the examples to show the non-vanishing come from products there remains the possibility that $\nu^*(p_1^2 - 2p_2)$ is 0. Thus, in the lowest degree in which this question is open it becomes:

F.2.6 Question : *Does there exist a codimension 4 transversely oriented Haefliger structure such that* $\nu^*(p_1^2 - 2p_2)$ *is not trivial rationally ?*

F.2.7. Question : *Does there exist a codimension 4 transversely oriented Haefliger structure on* S^8 *with non-trivial second Pontrjagin class for its normal bundle ?*

F.3. Paul Schweitzer

Godbillon's last paper [17] constructs a universal characteristic complex F_q for characteristic classes of codimension q foliations with trivialized normal bundle. For any manifold M with such a (smooth) foliation $\mathcal{F}$, with trivialisation, there are cochain maps :

$$C^*(M_q) \xrightarrow{\Phi} F_q \xrightarrow{\alpha_q} A^*(M)$$

where $C^*(M_q)$ is the Gelfand-Fuchs cochain complex and $A^*(M)$ is the de Rham complex of M.

F.3.1. Question : *Is the map on homology* $H(\Phi)$ *injective ? surjective ?*

Remark : If it is not surjective, there may be new characteristic classes. Godbillon shows that $H(\Phi)$ is an isomorphism for $q = 1, 2$.

F.4. Takashi Tsuboi

F.4.1. Question : *Construct a smooth codimension 1 foliation* $\mathcal{F}$ of $S^3 \times S^3$ *such that its Godbillon Vey invariant* $GV(\mathcal{F}) = (a, b) \in H^3(S^3 \times S^3)$ *with* a/b *being irrational.*

it Remark The construction of "discontinuous invariants" has been greatly generalized in the article "characteristic classes for flat bundles and their formulas", by J. Dupont, [12].

F.5. Steve Hurder

We assume that V is a compact manifold without boundary, and $\mathcal{F}$ is a codimension q foliation with transverse differentiability class C^r.

Transverse Euler classes

Assume that $\mathcal{F}$ is C^1, has even codimension $q = 2m$, and both $T\mathcal{F}$ and the normal bundle $\mathcal{Q} \to V$ are oriented. Let $\chi(\mathcal{Q}) \in H^q(V;\mathbf{R})$ denote the Euler class of $\mathcal{Q}$. Let μ be a holonomy-invariant transverse measure for $\mathcal{F}$, and $[C_\mu] \in H_p(V;\mathbf{R})$ the Ruelle-Sullivan class associated to μ. We say that an invariant transverse measure μ for $\mathcal{F}$ is *without atoms* if $\mu(\delta_L) = 0$ for each compact leaf $L \subset V$ and associated transversal δ_L.

F.5.1. Problem. *Show that the average transverse Euler class $\chi(\mathcal{Q}) \cap [C_\mu] \in H_{p-q}(V;\mathbf{R})$ vanishes whenever μ is without atoms.*

For example, it is an open problem to show that if $\mathcal{F}$ has no compact leaves then $\chi(\mathcal{Q}) \cap [C_\mu] = 0$.

Special cases of the problem **F.5.1.** have been established in [30, 31]. An embedded closed submanifold $N \subset V$ is said to be *almost tangent* to $\mathcal{F}$ if there exists an $\varepsilon > 0$ such that the restricted exponential map $Q_{\epsilon|N} : N \to V$ is a diffeomorphism onto an open tubular neighborhood $U_\epsilon \supset N$, and sends the fibers of $\mathcal{Q}_\epsilon \mid N \to N$ to discs transversal to the foliation $\mathcal{F}$. A transverse measure μ is said to be *almost compact* if the closed support of μ is contained in such a tubular neighborhood $\pi : U_\epsilon \to N$ of an *almost tangent* closed submanifold $N \subset V$. (We say that the measure μ *covers* N).

F.5.2. Theorem (Theorem 1.10, [30]) *Let μ be an almost compact invariant transverse measure which covers an oriented submanifold $N \subset V$. Assume either that μ is not atomic, or μ is atomic with support on a compact leaf $L \subset U \to N$ which multiply covers the submanifold N under the projection π. Then $\chi(\mathcal{Q}) \cap [C_\mu] = 0$*

Homotopy theory of $\mathbf{B\Gamma^q}$

Let $B\Gamma_q$ (respectively, $B\Gamma_q^+$) denote the Haefliger classifying space of C^2 codimension-q foliations (respectively, with orientable normal bundle). Assume that $\mathcal{F}$ is C^2 with even codimension $q = 2m$, and the normal bundle $\mathcal{Q} \to V$ is oriented. The square of the transverse Euler class $\chi(\mathcal{Q})^2 = P_m(\mathcal{Q}) \in H^{2q}(V;\mathbf{R})$, so the Bott Vanishing Theorem implies that $\chi(\mathcal{Q})^4 = P_m(\mathcal{Q})^2 = 0$. However, the following remains unsolved :

F.5.3. Problem. *Does there exists a C^2-foliation $\mathcal{F}$ for which $\chi(\mathcal{Q})^3 \neq 0$?*

The only known method for attacking the problem is to try to directly construct a foliation with $\chi(\mathcal{Q})^3 \neq 0$. Towards this goal, there is a more general problem, whose solution would be extremely important for understanding the homotopy theory of $B\Gamma^q$. Let $\mathbf{SO(q)}$ denote the special orthogonal group in dimension q, and form the non-compact (trivial) fibration $\pi : V = \mathbf{S0(q)} \times \mathbf{R}^q \to \mathbf{S0(q)}$. Define two foliations on V : $\mathcal{F}_0$ is the product foliation with leaves $L_0(\vec{x}) = \mathbf{S0(q)} \times \{\vec{x}\}$ for $\vec{x} \in \mathbf{R}^q$, and $\mathcal{F}_1$ is the twisted foliation with leaves $L_1(\vec{x}) = \{(A, A\vec{x}) \mid A \in \mathbf{S0(q)}\}$ for $\vec{x} \in \mathbf{R}^q$. There is a diffeomorphism of V which carries the leaves of $\mathcal{F}_0$ to the leaves of $\mathcal{F}_1$.

F.5.4. Problem. *Construct a C^2, codimension-q foliation $\mathcal{F}$ on $V \times [0,1]$ which is everywhere transverse to the fibers of $V \times [0,1] \to \mathbf{S0(q)} \times [0,1]$, and restricts to $\mathcal{F}_t$ on $V \times \{t\}$ for $t = 0, 1$.*

The existence of such a foliation is known (abstractly) for $q \leq 4$. Even in low dimensions it is an open problem to produce the concordance$\mathcal{F}$ by an explicit process (cf. **Problem F.2.2** above by Vogt). The construction of such a foliation $\mathcal{F}$ is equivalent to proving the existence of a lifting $\tilde{g}$ of the adjoint g of the natural map $\mathbf{SO(q)} \to \Omega B\mathbf{SO(q)}$ in the diagram:

$$\begin{array}{ccc} & & B\Gamma_q^+ \\ & \overset{\tilde{g}}{\nearrow} & \downarrow \nu \\ \Sigma SO(q) - & \overset{g}{\longrightarrow} & BSO(q) \end{array}$$

A solution to Problem **F.5.3.** has two immediate corollaries. First, it yields a solution to **Problem F.2.5** by using that the normal bundle map $B\Gamma_q^+ \to B\mathbf{S0(q)}$ is natural with respect to products, and using the product formulas in cohomology. Also, it gives a solution to the following conjecture made in [27]:

F.5.5. Conjecture: $\Omega B\Gamma_q^+ \simeq \mathbf{S0(q)} \times \Omega F\Gamma_q$ *for all* $q \geq 1$.

This is deduced from a solution to Problem **F.5.3** by considering the Puppe sequence for the map $B\Gamma_q^+ \to B\mathbf{SO(q)}$. This conjecture is proven for $q \leq 4$ in [27], using known facts about the homotopy type of $B\Gamma_q^+$.

There are still many remaining open questions about the non-triviality of the secondary characteristic classes. The most extensive results on their non-triviality, to date, are described in the papers [21, 23]. We mention just one outstanding question in this direction. Recall that the cohomology groups $H^*(W0_q)$ are the secondary classes for C^2, codimension-q foliations (with no other assumption on their normal bundles) and there is a natural map $\Delta : H^*(W0_q) \to H^*(B\Gamma_q)$. There is an algebra map $W0_{q+1} \to W0_q$, and the *rigid classes* are those in the image of the induced map $H^*(W0_{q+1}) \to H^*(W0_q)$ (cf. section 5.43 of [33]).

F.5.6. Problem. *Show that the images of the rigid secondary classes vanish in* $H^*(B\Gamma_q)$*. That is, show that the composition*

$$H^*(W0_{q+1}) \longrightarrow H^*(W0_q) \xrightarrow{\Delta} H^*(B\Gamma_q)$$

is the trivial map.

Secondary characteristic classes and dynamics

Here are four open problems concerning how secondary classes are related to the geometry and dynamics of foliations.

F.5.7. Problem. *Let* $\mathcal{F}_0$ *and* $\mathcal{F}_1$ *be* C^2 *foliations on a compact manifold* V*, with secondary characteristic class maps* $\Delta_i : H^*(W0_q) \to H^*(V; \mathbf{R})$*. Assume there is a homeomorphism* $H : V \to V$ *which maps the leaves of* $\mathcal{F}_0$ *to the leaves of* $\mathcal{F}_1$*. Show that* $H^* \circ \Delta_1 = \Delta_0$*. That is, show that the secondary characteristic classes of a* C^2*-foliation are topological invariants.*

For codimension one, if the conjugacy H and its inverse are both *absolutely continuous*, then the problem has been solved (Theorem 3.10, [29] ; [51]).

The existence of a topological conjugacy H between $\mathcal{F}_0$ and $\mathcal{F}_1$ means that the two foliations have the same topological dynamics. However, topological conjugacy seems to imply very little about the differentiable conjugacy of the foliations. Positive results in this direction are based on properties of hyperbolic dynamical systems (cf. [16]; Theorem 2.15, [26]; [34]). There are elementary examples of topologically conjugate hyperbolic systems which are not smoothly conjugate (cf. [2, 38]) so that additional properties beyond hyperbolicity are required to produce relations between the smooth structures of topologically conjugate foliations. However, one conjectures that the existence of a non-trivial secondary invariant implies the existence of enough such hyperbolic and additional "combinatorial" structures to obtain useful differentiable information (cf. [16, 28]) which can be used to relate their secondary classes.

There are counter-examples to topological invariance of the Godbillon-Vey class if the foliations are not C^2. Ghys defined in [15] a "Godbillon-Vey" type invariant for piecewise C^2-foliations in codimension one, and then showed via surgery on Anosov flows on 3-manifolds that there are homeomorphic piecewise C^2-foliations with distinct "Godbillon-Vey" invariants. In another direction, Hurder and Katok defined in [29] a "Godbillon-Vey" type invariant for the weak-stable foliations of volume preserving Anosov flows on 3-manifolds, and showed that for the geodesic flow of a metric of variable negative curvature on a compact Riemann surface, the "Godbillon-Vey" invariant varies continuously and non-trivially as a function of the metric (Corollary 3.12, [H-K2]). The weak-stable foliations of all of these metrics are topologically conjugate. Both approaches to extending the Godbillon-Vey invariant are combined in work of Tsuboi [51].

The space $H^*(W0_q)$ is spanned by monomials of the form $\{y_I \wedge c_J\}$ where $y_I = y_{i_1} \wedge ...y_{i_l}$ for $I = (i_1 < ... < y_l)$ with each $1 \leq i_k \leq q$ an odd integer. The generalized Godbillon-Vey classes are those of the form $y_1 \wedge c_J$ where c_J has degree $2q$.

A foliation $\mathcal{F}$ is *amenable* if the Lebesgue measurable equivalence relation it defines on $V \times V$ is amenable in the sense of Zimmer [57]. Hurder and Katok showed in [28] that if $y_I \wedge c_J$ is *not* a generalized Godbillon-Vey class, then $\Delta(y_I \wedge c_J) = 0$ for an amenable foliation. The Roussarie examples, given by the weak-stable foliation of the geodesic flow for a metric of constant negative curvature, are amenable and have $\Delta(y_1 \wedge c_J) \neq 0$, so this result does not extend to all of the classes.

Here are three open questions about the generalized Godbillon-Vey classes.

F.5.8. Problem. *Suppose that Γ is a finitely generated amenable group and $\alpha : \Gamma \to \text{Diff}^{(2)}(N)$ is a C^2-action on a compact manifold N without boundary. Let M be a compact manifold whose fundamental group maps onto Γ, $\pi(M, m_0) \to \Gamma$, with associated normal covering $\tilde{M}$. Form the suspension foliation $\mathcal{F}_\alpha$ on the manifold*

$$V = M_\alpha = (\tilde{M} \times N)/\{(\gamma x, y) \sim (x, \alpha(\gamma)y) \text{ for } \gamma \in \Gamma.\}.$$

Show that each generalized Godbillon-Vey class $\Delta(y_1 \wedge c_J) \in H^{2q+1}(V; \mathbf{R})$ must vanish for $\mathcal{F}_\alpha$.

Remark : If the foliation $\mathcal{F}$ admits a homology invariant transverse measure which is *good* (i.e., positive on open transversals) and absolutely continuous, then $\Delta(y_1 \wedge c_J) = 0$ by results of either [22] or [28]. The hypothesis that Γ is amenable implies that there is a good invariant measure for the action on N, and hence also for the suspension foliation $\mathcal{F}_\alpha$. The point of the problem is to remove the hypothesis that the measure be absolutely continuous from the known vanishing result.

Let V be a Riemannian manifold, and $\mathcal{F}$ a C^2-foliation on V. Each leaf $L \subset V$ inherits a Riemannian metric from TV, and hence a Riemannian distance function determined by the path-length metric in L. Use this to define the ball $B_L(x, r) \subset L$ of radius $r > 0$ about a point $x \in L$. We say that L has sub-exponential growth if

$$\limsup_{r \to \infty} \frac{\log \{\mathrm{Vol}_L(B_L(x,r))\}}{r} = 0,$$

non-exponential growth if

$$\liminf_{r \to \infty} \frac{\log \{\mathrm{Vol}_L(B_L(x,r))\}}{r} = 0,$$

and exponential growth otherwise.

F.5.9. Problem. *Show that if almost every leaf of $\mathcal{F}$ has non-exponential growth, then every generalized Godbillon-Vey class $\Delta(y_1 \wedge c_J) \in H^{2q+1}(V; \mathbf{R})$ vanishes for $\mathcal{F}$.*

Remarks : The conclusion $\Delta(y_1 \wedge c_J) = 0$ was proven in [22] when almost every leaf has sub-exponential growth. For codimension one, Duminy proved (cf. [8]) that $\Delta(y_1 \wedge c_1) \neq 0$ implies that there exists an exceptional minimal set of positive measure, and hence the set of leaves with exponential growth also has positive measure. One expects that a similar phenomenon occurs in higher codimensions, so that when almost every leaf has non-exponential growth, the generalized Godbillon-Vey classes must vanish.

Two fundamental ideas appear in Duminy's study of the Godbillon-Vey invariant for C^2, codimension-one foliations: localization and the concept of "thickness". Localization was generalized to higher codimensions and other secondary classes as the basis for the Weil measures of [20], while thickness remains more of a mystery.

F.5.10. Problem. *Develop a concept of "thickness" for foliations of codimension $q > 1$ parallel to Duminy's definition for $q = 1$. Estimate the Weil measures in terms of thickness, and relate thickness to the dynamics of $\mathcal{F}$ in higher codimensions.*

G Riemannian geometry of foliations

G.1. Pawel Walczak

Let $\mathcal{F}$ be a transversally oriented, codimension 1 foliation of a compact manifold M. Given $\mathcal{F}$, what are the functions on M which can be the mean curvature of the foliation? More precisely: Let $N_{\max}$ be a maximal Novikov component, and $N_{\min}$ be a minimal Novikov component. Then

$$(-1)^{n+1} \int_{N_{\max}} H dv = vol \partial N_{\max} > 0$$

$$(-1)^{n+1} \int_{N_{\min}} H dv = vol \partial N_{\min} < 0$$

where H is the mean curvature function of $\mathcal{F}$ with respect to a Riemannian metric g on M. Let us define admissible functions using the above inequalities :

$$Adm(\mathcal{F}) = \left\{ \begin{array}{l} f : M \to \mathbf{R} \mid (-1)^{n+1} f(x) > 0 \text{ somewhere in any } N_{\max} \\ \\ \text{and } \ (-1)^{n+1} f(y) < 0 \ \text{ somewhere in any } \ N_{\min} \end{array} \right\}$$

and let us define the set $Mean(\mathcal{F})$ by

$$Mean(\mathcal{F}) = \left\{ \begin{array}{c} f : M \to \mathbf{R} \mid \exists g, \text{a Riemannian structure on } M \text{ such that} \\ f = H \text{ is the mean curvature of } \mathcal{F} \text{ with respect to } g \end{array} \right\}.$$

Obviously Mean $(\mathcal{F}) \ \subset$ Adm $(\mathcal{F})$.

Question : *Is it always true that Mean* $(\mathcal{F}) =$ Adm $(\mathcal{F})$?

G.2. Rémi Langevin (after a discussion with Paul Schweitzer).

Let $\mathcal{F}$ be a codimension 1 foliation of a sphere S^n. The sphere has its canonical curvature 1 metric, and at each point the determinant K of the second fundamental form of the leaf through the point is well defined. Let us call $\int_M \mid K \mid$ the total curvature of the foliation. A foliation minimizing the total curvature is called *tight*. Langevin proved in [36] that S^3 does not admit a tight foliation, using the existence of a Reeb component.

G.2.1. Question : *Prove (directly !) that* S^n *does not admit a tight foliation for all* n ?

H Other topics

H.1. José Luis Arraut

Let M describe a closed connected orientable m-dimensional smooth manifold, $\mathcal{F}$ a n-dimensional smooth foliation of M and $H^1(M, \mathcal{F})$ the 1^{st}-cohomology group of the foliated manifold $(M, \mathcal{F})$.

Definition. The co-rank $(M, \mathcal{F}) = p$, $0 \leq p \leq n \leq m$, if and only if $H^1(M, \mathcal{F})$ has elements $[\xi_1], ..., [\xi_p]$ with representative 1-forms $\xi_1, ..., \xi_p$, which are linearly independent at each point and p is maximal for this property.

H.1.1. Problem. *Characterize the foliated manifolds $(M, \mathcal{F})$ of co-rank p for fixed m and n.*
H.1.2. Problem. *Given M and n, characterize the foliations of co-rank p.*

Known Cases
I. Assume $m = n$, i.e. $\mathcal{F}$ is the foliation whose only leaf is M. It is proved in [49] that co-rank $(M) = p \geq 1$ if and only if M is a fiber bundle over T^p.
II. Assume $m = 3$ and $n = 2$ and that co-rank $(M, \mathcal{F}) = 2$. It is shown in [41] that M is a T^2-bundle over T^1. In [10] several necessary conditions on $\mathcal{F}$ are found and in [3] the solution to problem H.1.1 is completed for this case.

The first non-trivial open case in problem **C.4.1.** is for $m = 3,\ n = 2$ and $p = 1$.

H.2. Rémi Langevin

The question has been asked in different forms for years by René Thom.

H.2.1. Question : *What should be a Morse theory for foliations ?*

Remarks : Let M be a compact manifold, and let $\mathcal{F}$ be a foliation of M. Let also f be a Morse function on M such that the set

$$\Gamma(\mathcal{F}, f) = \left\{ \begin{array}{l} \text{critical points of the restriction} \\ \text{of } f \text{ to the leaves of } \mathcal{F} \end{array} \right\}$$

is almost everywhere a curve. R. Thom proved [46] that this condition is generic.

Let $M_c = f^{-1}((-\infty, c])$. Thom's question can be stated as: understand how $\mathcal{F}_{|M_c}$ changes when c increases. In particular there may exist a value $c^*(\mathcal{F})$ such that:

$$\begin{cases} \mathcal{F}_{|M_c} c < c^*(\mathcal{F}) \text{ has} \quad \text{no recurrent leaf} \\ \\ \mathcal{F}_{|M_c} c > c^*(\mathcal{F}) \text{ has a recurrent leaf} \end{cases}$$

Assuming that $\mathcal{F}_{|M_{c^*(\mathcal{F})}}$ has already a recurrent leaf L_{c^*} (maybe with pieces like $(x^2 - y^2) \geq 0$), does the sequence of critical values of $f_{|L_{c^*}}$ converge (generically?) to c^* showing a Feigenbaum-like phenomenon? What should the genericity condition be?

In the case of an irrational foliation of T^2 a similar question has been solved by S. Blank (again after an avatar of Thom's question).

Remarks : Two recent works which have made significant progress on Morse theory for foliations. The paper by Alvarez-Lopez [1] developed Morse theory for pseudo-groups. The paper by A. Connes and T. Fack [11] related foliation Morse theory and leafwise Betti numbers. Rephrasing a remark by R. Thom (as far as I understand it) there is a troublesome difference between the foliation case and Feigenbaum's. Comparing the tree of periodic points in Feigenbaum's case and the tree of connected components of $F^{-1}(L_{C^*})$ the critical values on different branches of the latter tree do not match as they do in Feigenbaum's case. Not to mention that Thom's tree seems to consist of roots rather than branches; and there are infinitely many of them.

H.3. L. Conlon.

Problem on Thurston norm of depth 1 leaves.

Let $\mathcal{F}$ be a transversely orientable taut foliation by surfaces of a compact, orientable 3-manifold M, $\partial M \neq \phi$. Assume that $\mathcal{F}$ is tangent to ∂M and that each leaf $L \subset M^\circ$ accumulates only on ∂M. Equivalently, $\mathcal{F} \mid M^\circ$ fibers M° over S^1. Necessarily the fibers are noncompact with finitely many ends.

— genus $(L) = \infty \Leftrightarrow$ some component of ∂M has genus > 1.

— Via intersection products, view $[L] \in H^1(M, \mathbf{Z}) = H^1(M; \mathbf{Z})$.

— Via Poincaré duality, view $[L] \in H_2(M, \partial M; \mathbf{Z}) = A$.

— View $\Lambda \subset H_2(M, \partial M; \mathbf{R})$ as the integer lattice and let

$$x : H_2(M, \partial M; \mathbf{R}) \to \mathbf{R}^+$$

be the Thurston (pseudo) norm. This takes integer values on Λ, hence the unit ball $B_x \subset H_2(M, \partial M; \mathbf{R})$ is a possibly unbounded polyhedron.

If every component of ∂M has genus 1, displace these components slightly inward to be transverse to $\mathcal{F}$. This changes $\mathcal{F}$ to a fibration $\mathcal{F}' : M \to S^1$ transverse to ∂M. The fiber F determines $[F, \partial F] \in H_2(M, \partial M; \mathbf{Z})$ and $[F, \partial F] = [L]$. In this case, Thurston proves in [48]:

H.3.1. Theorem. *The ray $R \subset H_2(M, \partial M; \mathbf{R})$ out of 0 through $[F, \partial F]$ pierces the interior of a face of B_x.*

Furthermore, every nondivisible lattice point $[F', \partial F'] \in \wedge$ such that the ray R' from 0 through $[F', \partial F']$ pierces the interior of this same face corresponds to a fiber F' of a fibration $M \to S^1$ (hence to a leaf L' of a foliation $\mathcal{F}'$ having the above properties of $\mathcal{F}$).

Every class $[\omega] \in H^1(M^\circ; \mathbf{R}) = H_2(M, \partial M; \mathbf{R})$ reached by a ray through the interior of this face is represented by a closed, nonsingular 1-form on M° which blows up "nicely" at ∂M.

H.3.2. Problem. *Formulate and prove a suitable analog of this theorem for the cases in which at least one component of ∂M has genus > 1.*

H.3.3. Problem. *In Thurston's case, genus $(L) < \infty$ and $\chi[L] =$ genus (L). In the general case, define a kind of "core genus" of L and prove that this realizes $\chi[L]$. (Tautness of the foliation is essential).*

H.4. Yoshihiko Mitsumatsu

H.4.1. Problem. *Let $\mathcal{F}$ be a codimension q foliation of the manifold M. Let X be a leaf-preserving vector field, the singular set of which is a saturated codimension 1 submanifold N of M. Define a "1-sided" residue along N.*

Remark : In a note which is too long to be reproduced here, D. Lehmann proposes an answer to this question.

H.5. Takashi Tsuboi

H.5.1. Problem. *Let* $f : S^m \to (K = f(S^m)) \subset \mathbf{R}^n$ *be a smooth embedding. Assume that* $T_Y K \cap K = Y$ *for all* $Y \in K$. *Must* $K = f(S^m)$ *be unknotted? That is, is there an embedded disk* $D^{m+1} \to \mathbf{R}^n$ *such that its restriction to* ∂D^{m+1} *coincides with* f *?*

Remarks

— If $n = m + 1$ the assumption implies convexity and K is unknotted (For example, use Reeb's theorem to obtain an easier than via the Schonflies Lemma).
— If $m = 1$ and $n = 3$ the assumption means that there are no cross tangencies (a line which is tangent to the curve at one point and crosses it at another). See the preprint of J. J. Nuño Ballesteros and M. C. Romero Fuster [4]. K. Kuga showed me in this case that $K = f(S^1)$ is unknotted.
— There are trivial cases where all embeddings are unknotted ($m = 1, n = 4, 2(m + 1) \leq n$).
— The question is open for the other cases, $m = 2, n = 4, m = 3, n = 5$...
— If there is a knot $f : S^m \to S^{n-1}$ which is still knotted in $\mathbf{R}^n$, this gives a counter example. This might be possible for large m and n.

Remark : The following are the references for the problems cited above. Note also that there are two general reference sources specifically for foliation theory: the *Bibliographie* for Godbillon's treatise on geometric theory [18], and the *References* to Tondeur's introduction to the Riemannian theory [50].

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